SOLUTIONS BY
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Math 115 Final Exam
December 16 1998

Department of Mathematics
University of Michigan

Name: ______________________ Instructor: ______________________

Signature: ______________________ Section: ______________________

General instructions: This test consists of 13 questions on 12 pages (including this cover sheet), totalling 100 points. When the exam begins, please count all of the pages of the exam, make sure none of them are missing, and write your name on each page.

Please read the instructions on each individual problem carefully, and indicate answers as directed. Show all your work! On questions 8–13 you can only be given credit for your answers if you show how you got them. If you found an answer using your calculator, briefly indicate what you did. If you are basing your reasoning on a graph from your calculator, sketch the graph. Write legibly. Use units wherever appropriate.

Problem	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	6	
7	5	
8	10	
9	10	
10	8	
11	12	
12	12	
13	12	
Total	100	

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NO PARTIAL CREDIT SECTION. (Problems 1–7.) No explanation necessary; no need to show work.

1. (5 points) Suppose the continuous function $f(x)$ has the property that

$$\int_{-1}^{3} f(x)dx = 10, \qquad \int_{3}^{5} f(x)dx = -2 \ .$$

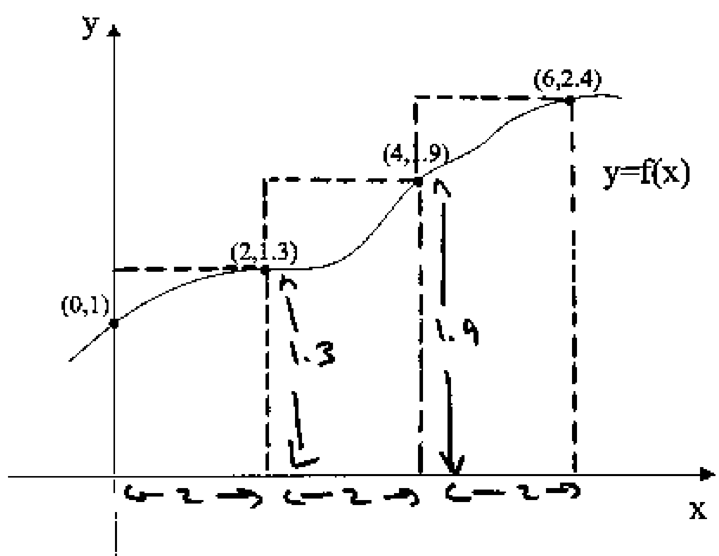
Find $\int_{-1}^{5} f(x)dx$.

$$\int_{-1}^{5} f(x)\,dx = \int_{-1}^{3} f(x)\,dx + \int_{3}^{5} f(x)\,dx$$

$$= 10 - 2 = 8$$

Answer: $\int_{-1}^{5} f(x)dx = \boxed{8}$

2. (5 points) The graph of $y = f(x)$ is given below.



Find the right hand sum approximation to $\int_0^6 f(x)dx$ for 3 equal subdivisions of the interval $0 \le x \le 6$.

$$RHS = 2[1.3 + 1.9 + 2.4] = 11.2$$

Answer: Right hand sum = $\boxed{11.2}$

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3. (5 points) At what point P on the graph of $f(x) = x^4$ is the tangent line parallel to the line $y = \frac{343}{2}x + 115$? Give your answer correct to one decimal place.

$f'(x) = 4x^3$. FIND P BY SOLVING

$4x^3 = \frac{343}{2} \Rightarrow x = \frac{7}{2} = 3.5$

$y = (3.5)^4 = 150.1$

Answer: $P = (\boxed{3.5}, \boxed{150.1})$

4. (5 points) A curve is given implicitly by the equation

$$3xy + y^3 = 2x^2.$$

Find $\frac{dy}{dx}$.

$3y + 3x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 4x$

$\Rightarrow (3x + 3y^2)\frac{dy}{dx} = 4x - 3y$

Answer: $\frac{dy}{dx} = \dfrac{\boxed{4x - 3y}}{\boxed{3x + 3y^2}}$

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5. (5 points) Let $f(x)$ be the function $f(x) = 4\cosh(3x)$. Find the derivative $f'(x)$ of the function $f(x)$.

$f'(x) = 12 \sinh(3x)$

Answer: $f'(x) =$ $\boxed{12 \sinh(3x)}$

6. (6 points) (*Attempting to do this problem using a calculator is time consuming and may not yield the correct answers*). You are given the following 8 functions labeled by the letters A through H.

$(A)\ 2^x - 40x^8$ $\quad (B)\ 20\log(x) + 18$ $\quad (C)\ \frac{3x^7}{2x^3-1}$ $\quad (D)\ x^5 + 6x^3$

$(E)\ 17x - 200$ $\quad (F)\ 3^x - 70x^{10}$ $\quad (G)\ 17x^2 + 8x$ $\quad (H)\ .01x^{1.5} + 100x$

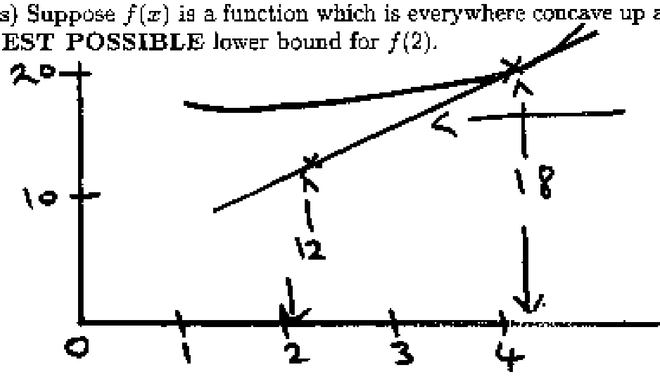
(a) (3 points) Write in the box the letter corresponding to the function which has largest values as $x \to +\infty$.

Answer: $\boxed{F}$ EXPONENTIAL GROWTH IS FASTEST

(b) (3 points) Write in the box the letter corresponding to the function which has smallest values as $x \to +\infty$.

Answer: $\boxed{B}$ LOGARITHMIC GROWTH IS SLOWEST

7. (5 points) Suppose $f(x)$ is a function which is everywhere concave up and satisfies $f(4) = 18$, $f'(4) = 3$. Find the **BEST POSSIBLE** lower bound for $f(2)$.



SLOPE OF LINE = 3

Answer: $f(2)$ must be larger than: $\boxed{12}$

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SHORT ANSWER SECTION. (Problems 8–13.) Limited partial credit may be possible, and a few sentences of explanation may be required. In addition, you may be asked to state your final answer in the form of a complete sentence. Remember to show your work!

8. (10 points) It is estimated that t years from now Santa will have

$$P(t) = \frac{6000}{1 + 3e^{-0.06t}}$$

reindeer to deliver all his presents.

(a) (6 points) How fast will Santa's reindeer population be growing 20 years from now? Give your answer correct to the nearest whole number. **Show all work** and state your answer in a complete sentence.

$$P'(t) = \frac{6000}{\left[1+3e^{-.06t}\right]^2} 3*(.06)e^{-.06t}$$

$$P'(20) = 89.77.$$

POPULATION WILL BE GROWING AT A RATE OF 90 REINDEER PER YEAR, 20 YEARS FROM NOW.

(b) (4 points) How many reindeer will Santa have in the long run? Give your answer correct to the nearest whole number. **Explain your reasoning** and state your answer in a complete sentence.

$$\lim_{t \to \infty} e^{-0.06t} = 0 \Rightarrow$$

$$\lim_{t \to \infty} P(t) = \frac{6000}{1+0} = 6000.$$

SANTA WILL HAVE 6000 REINDEER IN THE LONG RUN.

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9. (10 points) The height of the water level in feet on the sea wall in Bournemouth, England, t hours after midnight on November 26, 1998 is given by the trignometric function $h(t)$,

$$h(t) = 7 + 3\sin\left(\frac{t-1}{2}\right) \quad .$$

(a) (3 points) Find the height of the water level at low tide. **EXPLAIN** your answer.

SINE FUNCTION VARIES FROM -1 TO $+1$. HENCE MINIMUM VALUE OF h IS $7-3=4$

(b) (3 points) Find the first time after midnight that there is a high tide. Give your answer correct to the nearest minute and **SHOW** your work.

HIGH TIDE WHEN SINE $=1$.

$\Rightarrow \quad \frac{t-1}{2} = \frac{\pi}{2} \Rightarrow t = 1+\pi$

$= 4.1416$ HOURS AFTER MIDNIGHT

TIME $=$ 4:08 a.m.

(c) (4 points) The energy $E(t)$ of the tide, t hours after midnight is defined to be

$$E(t) = [h'(t)]^2 \quad .$$

Find the energy of the tide at 2:45 am. Do not include units and give your answer correct to three decimal places. **SHOW ALL WORK.**

$h'(t) = \frac{3}{2}\cos\left(\frac{t-1}{2}\right)$

ENERGY AT 2:45 a.m. IS

$[h'(2.75)]^2 = \left[1.5\cos\left(\frac{1.75}{2}\right)\right]^2$

$= .924$

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10. (8 points) According to Kepler's laws about the motion of the planets, the square of the length of a year of each planet in the solar system is proportional to the cube of the average distance of the planet from the sun. Here, the "length of a year of a planet" is the time needed for the planet to complete one full orbit around the sun.

The average distance from Jupiter to the sun is 5.203 times as large as the average distance from Earth to the sun. How many earth years does a year on Jupiter last? Give your answer correct to two decimal places.

Show all work and state your answer in a complete sentence.

D_E = DISTANCE FROM EARTH TO SUN.

D_J = DISTANCE FROM JUPITER TO SUN.

$$D_J = 5.203\, D_E.$$

L_E = LENGTH OF EARTH YEAR

L_J = LENGTH OF JUPITER YEAR.

$$L_E^2 = k\, D_E^3, \quad L_J^2 = k\, D_J^3$$

$$\frac{L_J^2}{L_E^2} = \frac{k\, D_J^3}{k\, D_E^3} = (5.203)^3$$

$$L_J = (5.203)^{3/2} L_E = 11.87\, L_E.$$

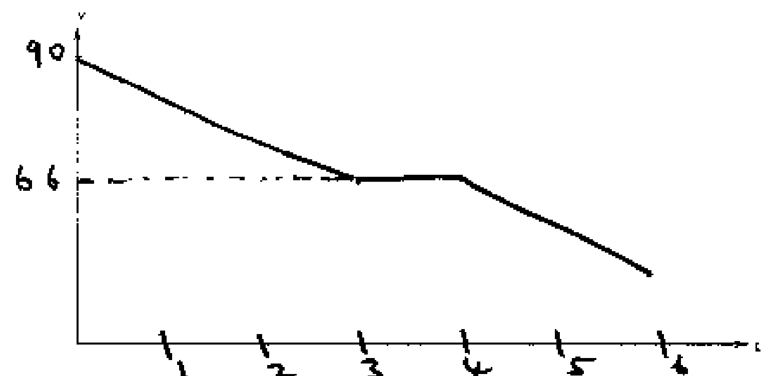
A JUPITER YEAR LASTS 11.87 EARTH YEARS.

11. (12 points) A car is travelling at a speed of 90 feet per second. The driver needs to come to a stop so he applies the brakes, resulting in a constant deceleration of 8 ft/sec^2 for the first 3 seconds. Then he leaves his foot off the brake for 1 second (giving zero deceleration). Finally he applies the brakes again, resulting in a constant deceleration of 11 ft/sec^2, until the car comes to a complete stop.

(a) (3 points) Let $v(t)$ be the velocity of the car in feet per second t seconds after the driver initially puts on the brakes. Fill in the blanks in the table below.

t	0	1	2	3	4	5	6
v(t)	90	82	74	66	66	55	44

(b) (3 points) On the axes provided below sketch a graph of $v(t)$ for $0 \leq t \leq 6$.



(c) (3 points) Find how long it took the car to come to a halt. **SHOW YOUR WORK** and give your answer in the form of a complete sentence.

t	6	7	8	9	10
v(t)	44	33	22	11	0

IT TAKES 10 SECONDS FOR THE CAR TO COME TO A HALT.

(d) (3 points) Find the total distance the car travelled after the driver initially put on the brakes. **SHOW YOUR WORK** and give your answer in the form of a complete sentence.

TOTAL DISTANCE IS AREA UNDER GRAPH IN (b)

1ST 3 SECS $= \frac{1}{2}(3)(90-66) + 3 \times 66$

$= 36 + 198 = 234$

4TH SEC $= 66$

LAST 6 SECS $= \frac{1}{2}(6)(66) = 198$

TOTAL $= 234 + 66 + 198 = 498$ ft.

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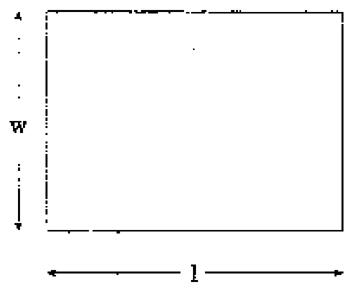
12. (12 points) Suppose you have some leftover paint and use it to mark the perimeter of a rectangular field and its midline on the ground (see figure). You have enough paint to draw lines of a total length of 500 meters, and you use up all the paint.

Let w and l denote the width and length of the field you mark, as shown in the figure below.



(a) (2 points) If you mark a field with length l equal to 190 m, what must the width w of the field be? In this case, what must the area A of the field be? (Remember that you use up all the paint.) **Show all work.**

$$w = \frac{1}{3}(500 - 2\cdot 190) = 40 \text{ m.}$$

$$A = 190(40) = 7600 \text{ m}^2$$

(b) (2 points) Now more generally, find a formula for the width w of the field in terms of its length l. **Show all work.**

$$3w + 2l = 500$$

$$w = \frac{500 - 2l}{3}$$

(c) (2 points) Find a formula for the area A of the field in terms of the length l only. **Show all work.**

$$A = lw = l\left[\frac{500-2l}{3}\right]$$

$$= \frac{500l}{3} - \frac{2l^2}{3}$$

(d) (6 points) Suppose you want to mark a field with the largest possible area. Find the length and the width of the field. **Show all work** and state your answer in a complete sentence.

$$A'(l) = \frac{500}{3} - \frac{4l}{3}$$

$$A'(l) = 0 \Rightarrow l = \frac{500}{4} = 125 \text{ m}$$

$$\Rightarrow w = \frac{500-2(125)}{3} = \frac{250}{3} \text{ m.}$$

$A'(l) > 0$ if $0 < l < 125$

$A'(l) < 0$ if $l > 125$.

Hence $l = 125$ gives MAXIMUM VALUE OF A.

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13. (12 points) A company which manufactures radios has a factory which can produce between 500 and 1000 radios per day. The factory's daily marginal cost function in dollars per radio, $MC(q)$, for producing q radios per day is given by the formula,

$$MC(q) = ln(500 + q), \quad 500 < q < 1000.$$

(a) (4 points) Suppose the company can sell a radio for \$7.00. If the company assumes it can sell all the radios it produces, how many radios per day should the factory manufacture to maximize the company's profit? Give your answer correct to the nearest whole number and **EXPLAIN** why it is correct.

$MC(q) = 7 \Rightarrow 500 + q = e^7$

$\Rightarrow q = e^7 - 500 = 597.$

PROFIT $P(q) = 7q - C(q)$

$P'(q) = 7 - MC(q)$.

FOR MAXIMUM PROFIT SOLVE

$P'(q) = 0 \Rightarrow q = 597.$ $P'(q) > 0$ IF

$q < 597$, $P'(q) < 0$ IF $q > 597 \Rightarrow$

$q = 597$ MAXIMUM

(b) (4 points) Let $C(q)$ be the factory's daily cost function in dollars for producing q radios per day. Write down an integral formula for $C(800) - C(700)$, and use your calculator program RSUMS to evaluate $C(800) - C(700)$ correct to the nearest dollar.

$C(800) - C(700) = \int_{700}^{800} MC(q)\, dq$

$= \int_{700}^{800} \ln(500 + q)\, dq = 713$

USING RSUMS

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(c) (4 points) Suppose the factory is currently manufacturing 700 radios per day and selling them at a price of \$6.50 each. If the factory were to increase production to 800 radios per day what would the minimum price be that the company needs to sell the radios at to increase its profit? **SHOW YOUR WORK** and give your answer correct to the nearest cent.

$$800p - C(800) > 700(6.5) - C(700)$$

$$\Rightarrow p > \frac{1}{800}\left[700(6.5) + C(800) - C(700)\right]$$

$$= \frac{1}{800}\left[700(6.5) + 713\right]$$

$$= 6.58.$$

COMPANY NEEDS TO CHARGE AT LEAST \$ 6.58.