SOLUTIONS by MARK KROSKY

Math 115 Calculus Exam I
October 1 1998

Department of Mathematics
University of Michigan

Name: ______________________ Instructor: ______________________

Signature: ______________________ Section: ______________________

General instructions: This test consists of 9 questions on 11 pages (including this cover sheet), totaling 100 points. When the exam begins, please count all of the pages of the exam, make sure none of them are missing, and write your name on each page.

Please read the instructions on each individual problem carefully, and indicate answers as directed. Show all your work! On questions 5-9 you can only be given credit for your answers if you show how you got them. If you found an answer using your calculator, briefly indicate what you did. If you are basing your reasoning on a graph from your calculator, sketch the graph. Write legibly. Use units wherever appropriate.

Problem	Points	Score
1	5	
2	10	
3	10	
4	12	
5	10	
6	12	
7	12	
8	12	
9	17	
Total	100	

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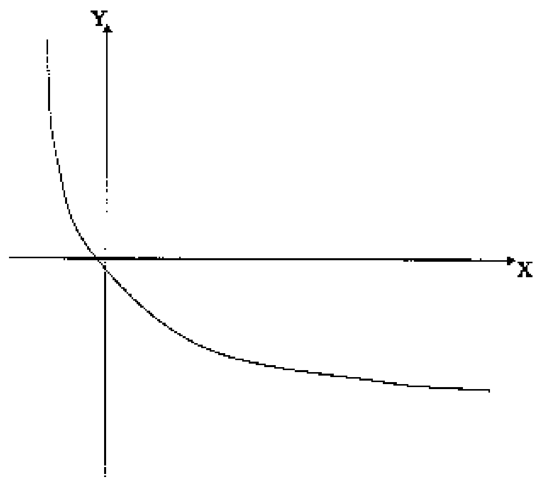
NO PARTIAL CREDIT SECTION. (Problems 1–4.) No explanation necessary; no need to show work.

1. (5 points) For the graph given below decide whether it is

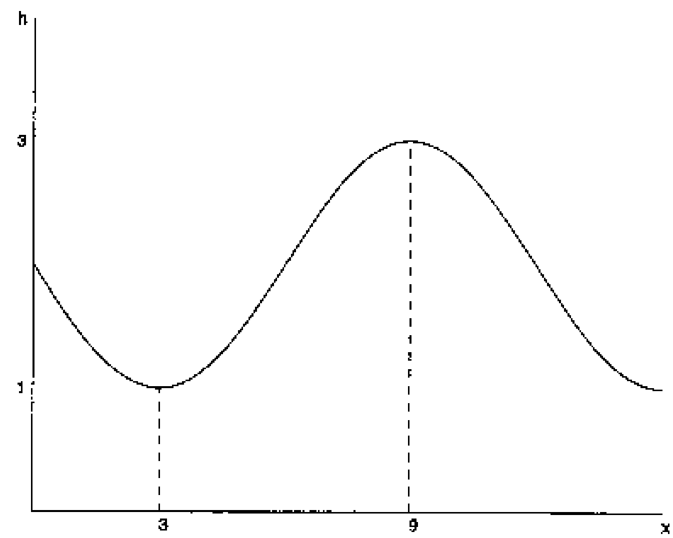
(a) increasing and concave up,
(b) increasing and concave down,
(c) decreasing and concave up, or
(d) decreasing and concave down.



Record the letter corresponding to the correct answer in the box provided below.

Answer: $\boxed{c}$

2. (10 points) A trigonometric function $h(x)$ has the graph given below. Find the amplitude, the period, and an explicit formula for $h(x)$.



Answers:

Amplitude:	1
Period:	12
formula for $h(x)$=	$-\sin\left(\frac{\pi}{6}\right)x + 2$

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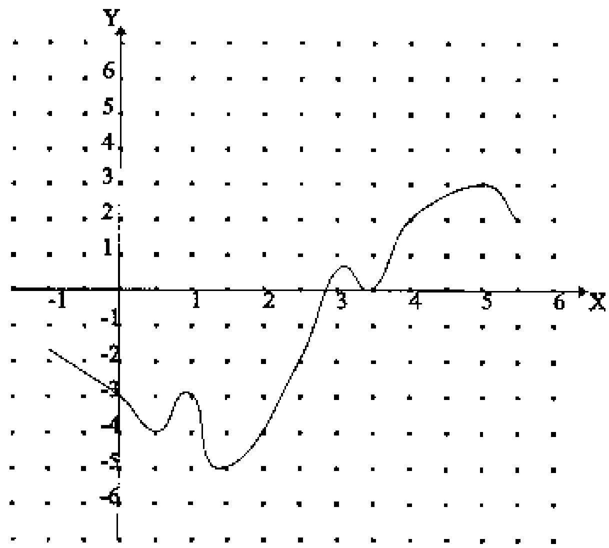
3. (10 points) A function f is given by the graph $y = f(x)$ below.



(a) Find $f(4)$ and $f(f(4))$ as accurately as you can.

Answers: $f(4) = \boxed{2}$

$f(f(4)) = \boxed{-4}$

(b) Find the range of the function. Give your answer as accurately as you can.

Answer: Range = $\boxed{-5 \leq y \leq 3}$

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4. (12 points) Tables of values for three functions f, g, h are given below.

x	f(x)
0	50
2	55
4	72
6	100

x	g(x)
5	10.0
8	8.8
11	7.6
14	6.4

+3, +3, +3; −1.2, −1.2, −1.2

x	h(x)
0	16.0
1	24.0
2	36.0
3	54.0

+1, +1, +1; $\times\frac{3}{2}$, $\times\frac{3}{2}$, $\times\frac{3}{2}$

(a) Which of the functions f, g, h is linear? Which is exponential?

Answers: Linear: $\boxed{g(x)}$

Exponential: $\boxed{h(x)}$

(b) Find a formula for the linear function.

$$\text{slope} = \frac{8.8 - 10.0}{8-5} = \frac{-1.2}{3} = -0.4$$

$g(x) = -0.4x + b$

Plug in a point: $10 = (-0.4)(5) + b$

$10 = -2 + b$

$b = 12$

Check: $(-0.4)(5) + 12 = 10$ ✓

$(-0.4)(8) + 12 = -3.2 + 12 = 8.8$ ✓

$(-0.4)(11) + 12 = -4.4 + 12 = 7.6$ ✓

$(-0.4)(14) + 12 = -5.6 + 12 = 6.4$ ✓

Answer: $\boxed{g(x) = -0.4x + 12}$

(c) Find a formula for the exponential function.

$h(x) = A \cdot B^x$

$h(0) = 16$, so $A = 16$

$h(1) = 16 \cdot B^1 = 24$

$B = \frac{24}{16} = \frac{3}{2}$

Check: $16 \cdot (\frac{3}{2})^0 = 16$ ✓

$16 \cdot (\frac{3}{2})^1 = 24$ ✓

$16 \cdot (\frac{3}{2})^2 = 36$ ✓

$16 \cdot (\frac{3}{2})^3 = 54$ ✓

Answer: $\boxed{h(x) = 16 \cdot \left(\frac{3}{2}\right)^x}$

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SHORT ANSWER SECTION. (Problems 5–8.) Limited partial credit may be possible, and a few sentences of explanation may be required. In addition, you may be asked to state your final answer in the form of a complete sentence. Remember to show your work!

5. (10 points) Kim and Louise have just started college. They both worked over the summer and saved some of their earnings. Kim saved \$2500 and Louise saved \$1500. Louise has just gotten a part time job working 12 hours per week at a rate of \$7 per hour. Kim does not have a job but spends her savings at a rate of \$45 per week. Louise spends \$50 per week. Find how many weeks it will take for Louise to have more money than Kim.

NO EXPLANATION NECESSARY, but show all your work, and state your final answer in the form of a complete sentence.

Let t = time in weeks since the end of the summer

$K(t)$ = Kim's savings in \$

$L(t)$ = Louise's savings in \$

$K(0) = 2500$, $L(0) = 1500$

Kim earns \$0/week, spends \$45/week. So, K is linear and the slope of K is -45.

Louise earns $\$12 \cdot 7 = \$84$/week, spends \$50/week. So, L is linear, and the slope of L is $84 - 50 = 34$.

$$\boxed{K(t) = -45t + 2500 \qquad L(t) = 34t + 1500}$$

When does $K(t) = L(t)$?

$$-45t + 2500 = 34t + 1500$$
$$1000 = 79t$$
$$t = \frac{1000}{79} = 12 + \frac{52}{79}$$

$$\begin{array}{r} 12 \\ 79\overline{)1000} \\ \underline{79} \\ 210 \\ \underline{158} \\ 52 \end{array}$$

Assuming that Louise gets paid at the end of the week and they spend their money on payday, it takes 13 weeks for Louise to have more money than Kim.

6. (12 points) Mark has just received the results of his first two exams at Ann Arbor University. He makes a table, given below, of the amount of work per week he dedicated to each course outside of class time, against his score (out of 100) on the exam.

Course	hours/week study	score (out of 100)
Chemistry	6	85
Economics	4	81.5

Mark wants to use this information to help him study more efficiently in future. He decides to model his data with a simple rational function. He chooses the function,

$$f(x) = 100\left(\frac{ax+6}{7x+12}\right) \quad .$$

Here $f(x)$ is the score on a 100 point exam Mark expects to get when he spends x hours per week in study for the course. Mark determines the value of the constant a from his data.

(a) (4 points) What score does Mark expect to get on an exam without doing any work outside class?

$f(0) = 100\left(\frac{a\cdot 0+6}{7\cdot 0+12}\right) = 100\cdot\frac{6}{12} = 50$ Mark expects to get a 50 not doing any work outside class.

(b) (4 points) Determine the value of the constant a Mark obtained from his data. **NO EXPLANATION NECESSARY**, but show all your work.

I plug in a point to solve for a.

Check: $81.5 = 100\left(\frac{6.65\cdot 4+6}{7\cdot 4+12}\right)$ ✓

$$85 = 100\left(\frac{a\cdot 6+6}{7\cdot 6+12}\right)$$

$$85 = 100\left(\frac{a\cdot 6+6}{54}\right)$$

$$85\cdot 54 = 100(a\cdot 6+6)$$

$$\frac{85\cdot 54}{100} = 6a+6$$

$$6a = \frac{85\cdot 54}{100} - 6$$

$$a = \frac{1}{6}\left(\frac{85\cdot 54}{100} - 6\right), \quad \boxed{a = 6.65}$$

(c) (4 points) Assuming there are an unlimited number of hours in a week, what is the maximum score Mark's model predicts he can get on an exam no matter how much he studies? **EXPLAIN** in no more than **TWO** sentences.

As $x \to \infty$, 6 is negligible compared with ax, and 12 is negligible compared with $7x$.

So, as $x \to \infty$, $f(x) \longrightarrow 100\cdot\frac{a\cancel{x}}{7\cancel{x}} = 100\cdot\frac{a}{7} = \boxed{95}$.

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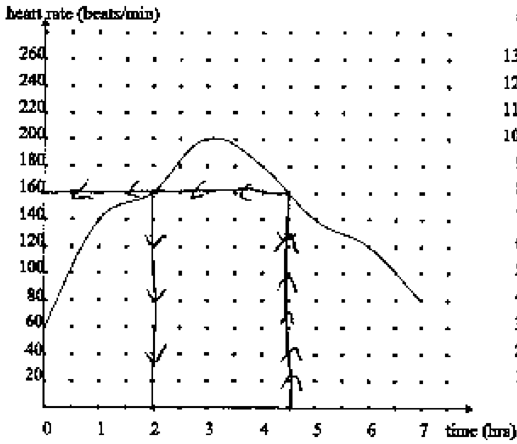
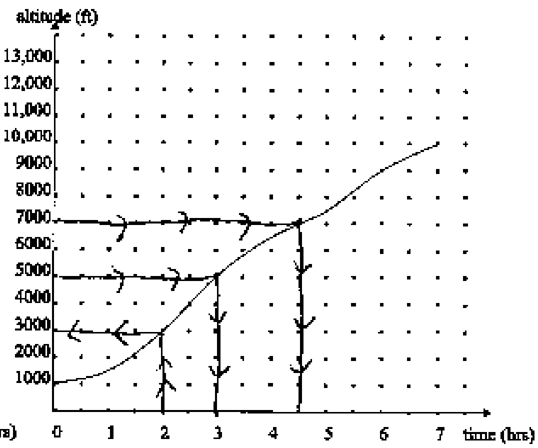
7. (12 points) Alex is climbing a mountain. He carries with him a device that monitors his heart rate and his altitude. Below are graphs of the data recorded.





(a) (4 points) Let $a(t)$ be the function given by the graph on the right. Solve $a(t) = 5000$ for t and **EXPLAIN** the practical meaning of your solution using the terms "altitude", "hours", and "feet".

Solving $a(t) = 5000$ ft yields the solution $t = 3$ hours.
This means that Alex was at ~~[illegible]~~ an altitude of 5000 feet after 3 hours.

(b) (3 points) How fast was Alex's heart beating when he reached an altitude of 7000 ft? State your answer in a complete sentence.

$a^{-1}(7000) = 4.5$ hours. Let $h(t)$ = the function represented by the graph on the left.
~~[illegible]~~ $h(4.5) = 160$
When Alex reached 7000 feet, his heart was beating at a rate of 160 beats/min.

(c) (5 points) Let $f(x)$ be the function that gives Alex's heart rate in beats per minute at an altitude of x feet. **EXPLAIN** why this function is or is not invertible. It is not enough to say "f does (or does not) pass the horizontal line test."

Observe that $f(x) = h(a^{-1}(x))$.
Alex's heart is beating at a rate of 160 beats/min after 2 hours and after 4.5 hours.
After ~~[illegible]~~ 2 hours, Alex is at a height of 3000 feet.
After 4.5 hours, Alex is at a height of 7000 feet.
Since the output 160 beats/min occurs for two different altitudes, 3000 ft and 7000 ft, f is not invertible.

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8. (12 points) A magician sets free 115 ghosts onto a wonderful island. The population of ghosts triples every two days.

(a) (6 points) If the ghost population reaches 999,999 then the wonderful island will become so unstable that it will slip into a parallel universe. How many full days are left before this happens (don't include fractions in your answer)? **SHOW ALL WORK** and state your answer in a complete sentence.

Let t = time in days since the ghosts were set free.
$g(t)$ = ghost population
$g(0) = 115$
$g(t) = A \cdot B^t$, $A = 115$
$g(2) = 345$
$\frac{g(2)}{g(0)} = \frac{345}{115} = 3 = \frac{\cancel{A} \cdot B^2}{\cancel{A} \cdot B^0}$
$B^2 = 3$, $B = \sqrt{3}$

So, $g(t) = 115 \cdot (\sqrt{3})^t$
I need to solve $g(t) = 999{,}999$
$999{,}999 = 115 \cdot (\sqrt{3})^t$
$\ln \frac{999999}{115} = t \ln \sqrt{3}$
$t = \ln\left(\frac{999999}{115}\right) / \ln \sqrt{3}$
$t = 16.51279...$

There are 16 full days left before the wonderful island slips into a parallel universe.

(b) (6 points) Inspired by her brother, the magician's sister sets free 1150 monsters on the island **THREE** days after the ghosts were set free . The monster population doubles every five days, and the island will become unstable and slip into the parallel universe if the monster population exceeds 8,888.

You know a spell that you can use on either the ghost population or the monster population that will stop that population from multiplying. Unfortunately the spell can only be used on one of the populations and only once.

To preserve this wonderful island as long as possible, which population should you use the spell on? **SHOW ALL WORK** and state your answer in a complete sentence.

Let t = time in days since the ghosts were set free.
$m(t)$ = monster population
$m(3) = 1150$
$m(8) = 2300$
$m(t) = A \cdot B^{t-3}$
$m(3) = A \cdot B^{3-3} = 1150$
$A = 1150$
$2 = \frac{m(8)}{m(3)} = \frac{\cancel{A} \cdot B^5}{\cancel{A} \cdot B^0} = B^5$
$B^5 = 2$
$B = (2)^{\frac{1}{5}}$

$m(t) = 1150 \cdot (2^{\frac{1}{5}})^{t-3}$
or $m(t) = 1150 \cdot (1.14869...)^{t-3}$
or $m(t) = 758.717... (1.14869...)^t$

Solve $m(t) = 8888$
$8888 = 1150 \cdot (2)^{\frac{1}{5}(t-3)}$
$\ln \frac{8888}{1150} = (t-3) \ln 2^{\frac{1}{5}}$
$\frac{\ln\left(\frac{8888}{1150}\right)}{\ln 2^{\frac{1}{5}}} + 3 = t$
$t = 17.751124...$

There are 17 full days before the island slips into a parallel universe due to the monsters, starting from when the ghosts were set free. Therefore, the spell should be used on the ghosts.

ESSAY QUESTION

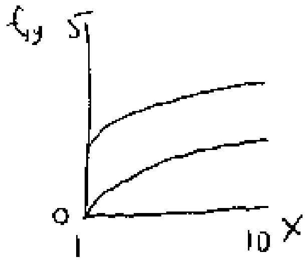
9. (17 points) Let f and g be the functions,

$$f(x) = \log(x), \quad g(x) = \log(100x), \quad 0 < x < \infty \ .$$

(a) (4 points) Use your calculator to complete the table below, recording the values correct to two decimal places.

x	f(x)	g(x)
0.4	-.40	1.60
1.8	.26	2.26
2.4	.38	2.38
3.7	.57	2.57

(b) (3 points) Graph both functions f and g on the same screen on your calculator using the window $1 < x < 10$, $0 < y < 5$. **DESCRIBE** in no more than **TWO** sentences what you see.



I see that the graph of $g(x)$ is the graph of $f(x)$ ~~[illegible]~~ translated up 2 units. I see that f and g are increasing and concave down.

(c) (4 points) What relationship between the functions f and g do your calculations in (a) and graphs in (b) suggest?

The relationship suggested is $g(x) = f(x) + 2$

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(d) (6 points) **EXPLAIN CAREFULLY**, step by step, using the laws of logarithms, how to obtain the relationship you found in (c).

$g(x) = \log 100x$

$g(x) = \log 100 + \log x$ because $\log ab = \log a + \log b$

$g(x) = \log 10^2 + \log x$ because $10^2 = 100$

$g(x) = 2 + \log x$ because $\log 10^a = a$

$g(x) = 2 + f(x)$ because $f(x) = \log x$