SOLUTIONS

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Math 115 Calculus Exam II
November 12 1998

Department of Mathematics
University of Michigan

Name: Key Instructor: ____________

Signature: ____________ Section: ____________

General instructions: This test consists of 10 questions on 10 pages (including this cover sheet), totalling 100 points. When the exam begins, please count all of the pages of the exam, make sure none of them are missing, and write your name on each page.

Please read the instructions on each individual problem carefully, and indicate answers as directed. Show **all** your work! On questions 6–10 you can only be given credit for your answers if you show how you got them. If you found an answer using your calculator, briefly indicate what you did. If you are basing your reasoning on a graph from your calculator, sketch the graph. Write legibly. Use units wherever appropriate.

Problem	Points	Score
1	6	
2	6	
3	12	
4	10	
5	12	
6	10	
7	10	
8	10	
9	12	
10	12	
Total	100	

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NO PARTIAL CREDIT SECTION. (Problems 1–5.) No explanation necessary; no need to show work.

1. (6 points) Find the equation of the tangent line to the graph of $f(x) = x^2 + 3x$ at the point $(1, 4)$.

$f'(x) = 2x + 3$ $\qquad\qquad$ $y = mx + b$

$f'(1) = 2 \cdot 1 + 3 = 5 = m$

$y - 4 = 5(x-1)$

$y = 5x - 5 + 4$

$y = 5x - 1$

Equation of the tangent line:

y= $\boxed{5}$ x + $\boxed{-1}$

2. (6 points) The table below contains some values of the function $f(x)$ and its derivative, $f'(x)$.

x	f(x)	$f'(x)$
0	2	5
1	4	-3
2	-2	3
3	4	2

Let $h(x)$ be the function $h(x) = [f(x)]^2$. Find the value of $h'(2)$.

$h'(x) = 2f(x) \cdot f'(x)$

$h'(2) = 2 \cdot f(2) \cdot f'(2)$

$= 2 \cdot (-2) \cdot 3$

$= -12$

$h'(2) = \boxed{-12}$

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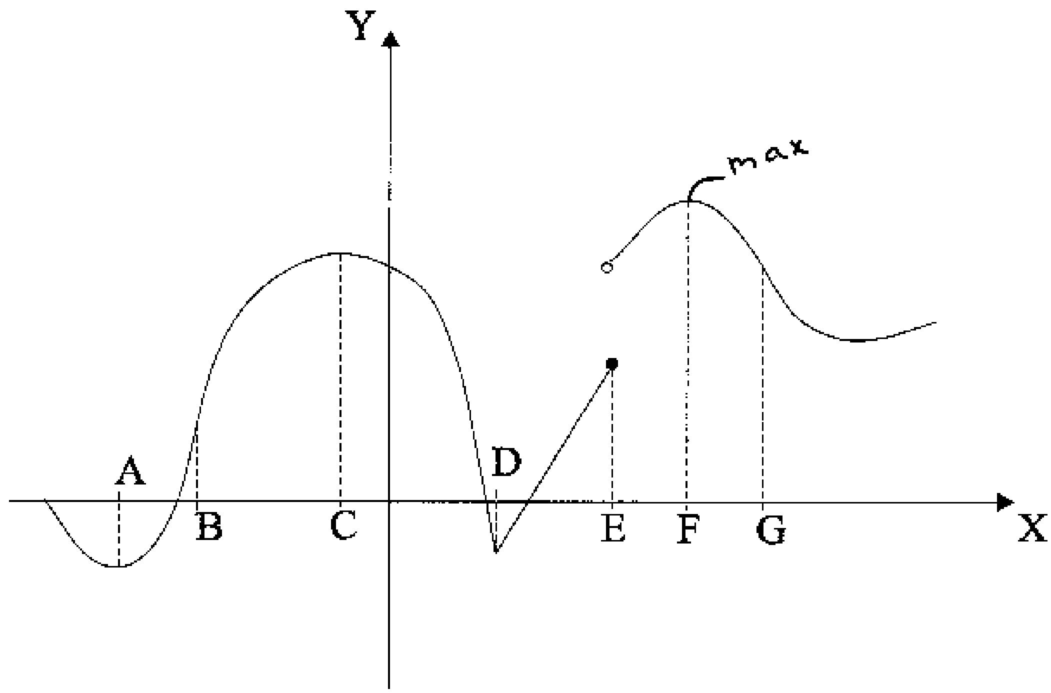
3. (12 points) Below is the graph of a function $y = f(x)$.



List the point A, B, C, D, E, F, G, at which

f has a global maximum:	F	
f' has a local maximum:	B	the slope of the tangent line is the greatest at B
$f' < 0$:	G	f is decreasing at G
f is not continuous:	E	

4. (10 points) A differentiable function f takes the values given in the table below.

x	1.5	1.6	1.7	1.8	1.9	2.0	2.1	2.2	2.3	2.4	2.5
f(x)	1.83	1.84	1.87	1.86	1.84	1.80	1.75	1.68	1.61	1.52	1.43

(a) (5 points) Find from the table the **BEST** approximation to the slope of the tangent line to the graph of $y = f(x)$ at $x = 2$.

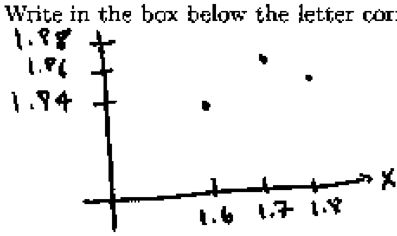
$$\text{slope} = f'(2) \approx \frac{f(2.1) - f(1.9)}{2.1 - 1.9} = \frac{1.75 - 1.84}{0.2} = -0.45$$

Slope of the tangent line at $x = 2$: $\boxed{-0.45}$

(b) (5 points) The **SMALLEST** interval in which the equation $f'(x) = 0$ is **GUARANTEED** to have a solution is one of the following:

(A) $1.6 < x < 1.7$,
(B) $1.7 < x < 1.9$,
(C) $1.6 < x < 1.9$,
(D) $1.5 < x < 1.8$,
(E) $1.5 < x < 2.5$,
(F) $1.6 < x < 1.8$,
(G) $1.5 < x < 1.7$,
(H) $1.7 < x < 1.8$.

Write in the box below the letter corresponding to the smallest interval.



Smallest interval corresponds to the letter: $\boxed{F}$

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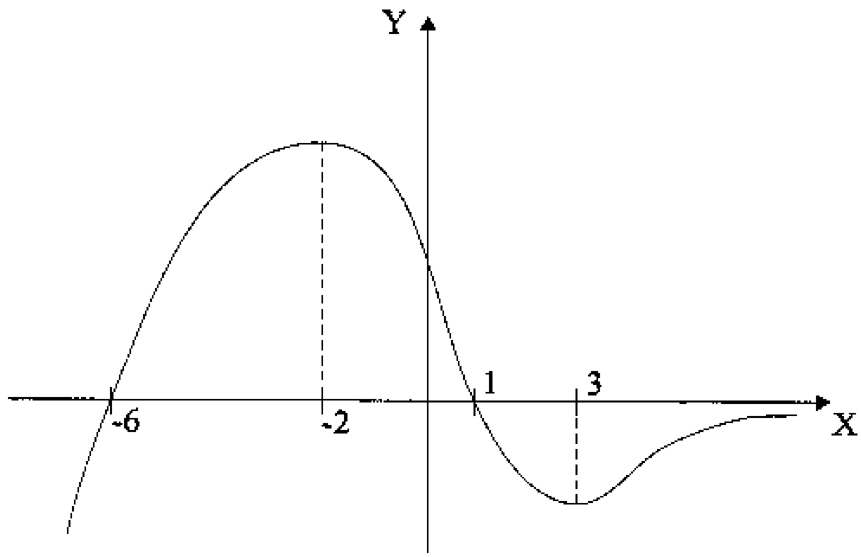
5. (12 points) The graph below is the graph $y = f''(x)$ of the **SECOND DERIVATIVE** of the function f.



(a) (4 points) Find the range of values of x for which the graph $y = f(x)$ of the function f (**NOT** the graph of f'') is concave up.

when is $f''(x) > 0$

Answer: $-6 < x < 1$

(b) (4 points) Find the value of x at which the derivative f' of the function f has a local minimum.

$f'(x)$ has a local minimum at $x = -6$

(c) (4 points) Find the maximum possible number of solutions to the equation $f(x) = 0$.

Assuming that the only zeros of f'' are at -6 & 1 and that f'' is continuous, we have that f changes concavity twice.

The equation $f(x) = 0$ has at most 4 solutions

SHORT ANSWER SECTION. (Problems 6–10.) Limited partial credit may be possible, and a few sentences of explanation may be required. In addition, you may be asked to state your final answer in the form of a complete sentence. Remember to show your work!

6. (10 points) An exponential model for the population $P(t)$ of Ann Arbor, t years from today, is given by

$$P(t) = 125,000e^{0.0228t}.$$

(a) (5 points) Find, according to the model, how many years it will take for the population to reach 150,000. Give your answer correct to the nearest year. **SHOW ALL WORK and state your answer in a complete sentence.**

$$150,000 = 125,000\, e^{.0228t}$$

$$\frac{6}{5} = e^{.0228t}$$

$$\ln\left(\frac{6}{5}\right) = \ln e^{.0228t} = .0228t$$

$$t = \frac{\ln\left(\frac{6}{5}\right)}{.0228} \approx 8$$

It will take approximately 8 years for the population to reach 150,000, according to the model.

(b) (5 points) Find, according to the model, how fast the population will be increasing exactly 4 years from today. Give your answer correct to **THREE** significant digits. **SHOW ALL WORK and state your answer in a complete sentence.**

$$P'(t) = (125,000)(.0228)e^{.0228t}$$

$$P'(4) = 3122 = 3.122 \times 10^3$$

The population will be increasing at a rate of 3122 people per year in exactly four years from today.

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7. (10 points) The diameter of a football depends upon the amount of air pumped into the ball. If V denotes the amount of air (in liters) pumped into the football, then the diameter D (in cm) of the football can be determined as a function of V according to the formula $D = f(V)$.

(a) (6 points) A football coach finds out that $f(8) = 30$ and $f'(8) = 1.1$. Estimate how much air he has to pump into a football so that the diameter of the football is approximately 32 cm? **SHOW ALL WORK and state your answer in a complete sentence.**

$$f(8+h) \approx f(8) + h f'(8) \quad \text{for small } h.$$

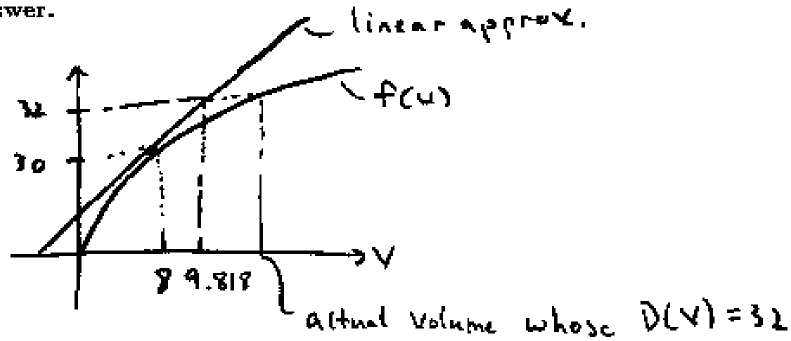
$$32 \approx 30 + h(1.1)$$

$$h \approx \frac{2}{1.1} \approx 1.818$$

$$8+h = 9.818$$

Using linear approximation, we estimate that 9.818 liters of air needs to be pumped into the football to make the diameter of the football 32 cm.

(b) (4 points) Suppose you know in addition that f'' is negative for all values of V. Given this information, is your answer to part (a) an overestimate, an underestimate, or can't you tell? **EXPLAIN your answer.**



actual volume > estimate.

Our estimate in part a) is an underestimate.

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8. (10 points) Ed is working for Ann Arbor's newest radio station and has to make a survey of the listening habits of U of M's calculus students.

The survey shows that the number of calculus students tuned in to the station t hours after 5:00 PM is

$$f(t) = -6t^3 + 81t^2 - 324t + 730.$$

(a) (2 points) Find, using your calculator, the time between 5:00 PM and midnight when the **MOST** calculus students are listening to the station. Find also the **NUMBER** of students listening at this time.

$t=0$. At 5:00pm , there are 730 students listening.

$f(0)=730$

(b) (2 points) Find, using your calculator, the time between 5:00 PM and midnight when the **LEAST** number of calculus students are listening to the station. Find also the **NUMBER** of students listening at this time.

$t=3$

$f(3)=325$

At 8:00pm , there are only 325 students listening.

(c) (6 points) **EXPLAIN**, using calculus, why the answers you obtained in parts (a) and (b) are correct. Your explanation must include a discussion of the **CRITICAL** points of the function f.

We first find the critical points

$$f'(t) = -18t^2 + 162t - 324$$
$$= -18(t^2 - 9t + 18)$$
$$= -18(t-6)(t-3)$$

$f'(t)=0 \Rightarrow t=3$ or $t=6$

The critical points are 3 & 6.

To find the global maximum & global minimum, we compare the values that f takes at the critical points & end points.

$f(3) = 325$ $\quad f(0) = 730$

$f(6) = 406$ $\quad f(7) =$ [redacted] 373.

Hence the global min occurs at $t=3$, (8:00pm)
and the global max occurs at $t=0$ (5:00pm)

9. (12 points) A family of curves is defined for each **POSITIVE** value of the constant a by the equation,

$$x^2 - 2x + ay^2 - 4ay + 3a + 1 = 0.$$

(a) (4 points) Using implicit differentiation, show that

$$\frac{dy}{dx} = \frac{1-x}{a(y-2)}.$$

$$\frac{d}{dx}\left(x^2 - 2x + ay^2 - 4ay + 3a + 1\right) = \frac{d}{dx}(0)$$

$$2x - 2 + 2ayy' - 4ay' = 0$$

$$2a(y-2)y' = 2(1-x)$$

$$y' = \frac{2(1-x)}{2a(y-2)}$$

$$\frac{dy}{dx} = \frac{1-x}{a(y-2)}$$

(b) (4 points) For what value of y does the curve have a vertical tangent line? **EXPLAIN** in no more than two sentences how you found your answer.

The slope of a vertical line is undefined, but $\frac{dx}{dy} = 0$ for a vertical line. Hence the curve has a vertical tangent line at $y = 2$ provided $x \neq 1$. We check that $(1,2)$ is not on the curve: $1 - 2 + 4a - 8a + 3a + 1 = 0 \Rightarrow a = 0$ a contradiction since $a > 0$. Thus $(1,2)$ is not on the curve.

(c) (4 points) For which value of the constant a does the graph have a vertical tangent line at $x = -2$? No explanation necessary, but **show all your work**.

$x = -2$, $y = 2$

$$(-2)^2 - 2(-2) + a2^2 - 4a(2) + 3a + 1 = 0$$

$$9 - a = 0$$

$$a = 9$$

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10. (12 points) Liquid is pumped through a cooling system at a rate of 3 gallons per minute. Since the system has some leaks the operator adds a small amount of chemical agent to the liquid in order to reduce the leakage. To test if the chemical agent is having an effect, the operator measures the amount of leakage 25 minutes after the agent has been added. He finds that the total amount of leakage after 25 minutes is 0.1 of a gallon and that the rate of leakage at this time is 0.002 gallons per minute.

(a) (2 points) Let $f(t)$ be the total amount of leakage in gallons during the t minutes after the agent was added. Find the values of $f(25)$ and $f'(25)$.

$$f(25) = 0.1$$

$$f'(25) = 0.002$$

(b) (3 points) Let $g(t)$ be the quotient of the total amount of leakage (in gallons) during the t minutes after the agent was added, to the the total amount of liquid (in gallons) which was pumped through the system during the same time interval. Find an expression for the function g in terms of the function f.

$$g(t) = \frac{f(t)}{3t}$$

(c) (4 points) Find, correct to **THREE** significant digits, the value of $g'(25)$. You must **SHOW** how you obtained your answer using the rules of calculus.

$$g'(t) = \frac{1}{3}\left(\frac{t f'(t) - f(t)}{t^2}\right)$$

$$g'(25) = \frac{1}{3}\left(\frac{25(.002) - (.1)}{25^2}\right) = -6.67 \times 10^{-4}$$

(d) (3 points)The operator decides that the agent works if the fraction of leakage to the total amount of liquid pumped through the system, decreases with time. Did the operator decide the chemical agent worked? **EXPLAIN**, showing all your calculations.

Since $g'(25) < 0$, the fraction of leakage to the total amount of liquid pumped through the system is decreasing with time for times near 25. So when the operator takes the measurements, he or she decides that the agent works.