Math 115 Calculus Final
April 24 1998

Department of Mathematics
University of Michigan

Name: ______________________ Instructor: ______________________

Signature: ______________________ Section: ______________________

General instructions: This test consists of 11 questions on 12 pages (including this cover sheet and a blank final page), totalling 100 points. When the exam begins, please count all of the pages of the exam, make sure none of them are missing, and write your name on each page.

Please read the instructions on each individual problem carefully, and indicate answers as directed. Show all your work! On questions 7–11 you can only be given credit for your answers if you show how you got them. If you found an answer using your calculator, briefly indicate what you did. If you are basing your reasoning on a graph from your calculator, sketch the graph. Write legibly. Use units wherever appropriate.

Problem	Points	Score
1	4	
2	4	
3	12	
4	4	
5	5	
6	5	
7	10	
8	12	
9	15	
10	15	
11	14	
Total	100	

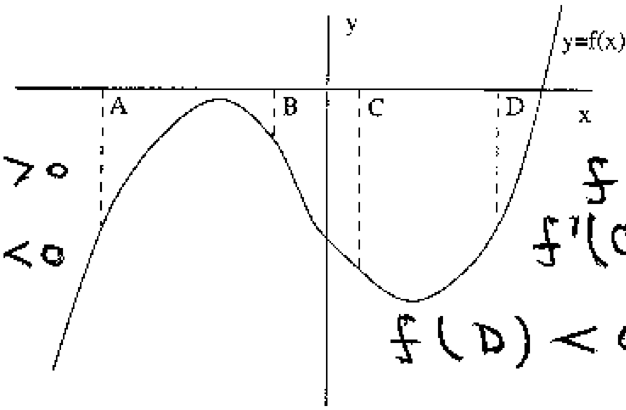
NO PARTIAL CREDIT SECTION. (Problems 1–6.) No explanation necessary; no need to show work.

1. (4 points) The graph below describes a function $f(x)$. At which of the labelled points, if any, do f, f' and f'' all have the same sign?

$f(A) < 0$, $f'(A) > 0$

$f(B) < 0$, $f'(B) < 0$

$f''(B) < 0$



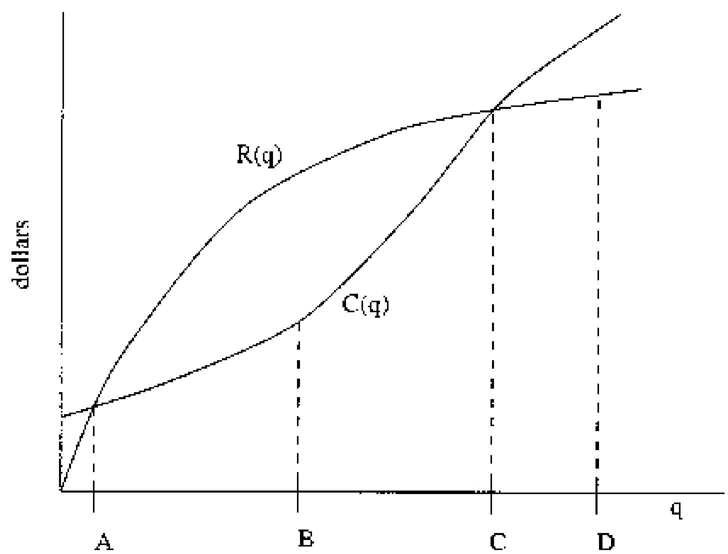
$f(C) < 0$,

$f'(C) < 0$, $f''(C) > 0$

$f(D) < 0$, $f'(D) > 0$

Answer: $\boxed{B}$

2. (4 points) A manufacturer reports that it costs her $C(q)$ dollars to produce q items and that she will receive $R(q)$ dollars in revenue from selling q items. If $C(q)$ and $R(q)$ are the functions in the graph below, at which of the marked production levels will her profits be largest?



Answer: $\boxed{B}$ profit $= R(q) - C(q)$

$R(q) - C(q)$ is largest at $q = B$.

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3. Suppose that $f(x)$ is a polynomial and you are told that $f(2) = -1$, $f'(2) = 3$ and $f''(2) = 5$.

(a) (4 points) Circle all of the statements below which are true.

f is increasing at 2 → $f'(2) = 3 > 0$	f is decreasing at 2
f is concave up at 2 → $f''(2) = 5 > 0$	f is concave down at 2
f has an inflection point at 2	f has a critical point at 2

(b) (5 points) Find the equation of the tangent line to $y = f(x)$ at the point where $x = 2$.

Equation of tangent line is

$$y - f(2) = f'(2)[x-2] \Rightarrow$$

$$y - (-1) = 3[x-2] \Rightarrow y = 3x - 7$$

Answer: $\boxed{y = 3x - 7}$

(c) (3 points) Use the tangent line approximation to estimate $f(1.9)$.

$$f(1.9) \simeq 3(1.9) - 7 = -1.3$$

$f(1.9) \approx \boxed{-1.3}$

4. (4 points) If $\int_1^5 g(x)dx = 5$ and $\int_1^2 g(x)dx = -1$, then evaluate $\int_2^5 g(x)dx$.

$$\int_1^2 g(x)\,dx + \int_2^5 g(x)\,dx = \int_1^5 g(x)\,dx$$

$$-1 + \int_2^5 g(x)\,dx = 5$$

$$\Rightarrow \int_2^5 g(x)\,dx = 6$$

$\int_2^5 g(x)dx = \boxed{6}$

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5. (5 points) Let the function $f(x)$ be described by the following table:

x	0	0.2	0.4	0.6	0.8
f(x)	0.2	0.3	0.5	0.7	0.9

Suppose that $h(x) = f(2x)$. Use the data in this table to give the best possible estimate of the value of $h'(0)$.

$$h'(x) = f'(2x)\frac{d}{dx}(2x)$$

$$= 2f'(2x)$$

$$\Rightarrow h'(0) = 2f'(0).$$

$$f'(0) \simeq \frac{f(.2)-f(0)}{.2-0} = \frac{.3-.2}{.2} = \frac{.1}{.2} = \frac{1}{2}$$

$h'(0) \approx$ $\boxed{1}$

6. (5 points) A particle is moving forward along a straight track. After t seconds it has covered

$$s = 60t - 32\ln(t+1) \text{ feet.}$$

What is its acceleration after 3 seconds? Remember to include units.

velocity $v = \frac{ds}{dt} = 60 - \frac{32}{t+1}$ ft/sec

acceleration $a = \frac{dv}{dt} = \frac{32}{(t+1)^2}$ ft/sec^2

$t = 3 \Rightarrow a = \frac{32}{16} = 2$ ft/sec^2

Answer: $\boxed{2 \text{ ft/sec}^2}$

SHORT ANSWER SECTION. (Problems 7–10.) Limited partial credit may be possible, and a few sentences of explanation may be required. You must show your work to receive credit.

7. (10 points) Suppose that Speedy Racer enters the final stretch of the Ann Arbor Grand Prix trailing her arch-rival Racer Yam. If she covers the remaining 380 feet in 3 seconds or less, then she will pass Racer Yam and win the race. She activates her secret boosters and her velocity increases and continues to increase until she crosses the finish line. The following table records her speeds at half second intervals after she activates her secret boosters.

time(sec)	0	0.5	1.0	1.5	2.0	2.5	3.0
velocity (ft/sec)	100	118	132	142	150	154	156

Did Speedy Racer win the race? Justify your answer in complete sentences. (If the data is inconclusive, explain carefully why it is inconclusive.)

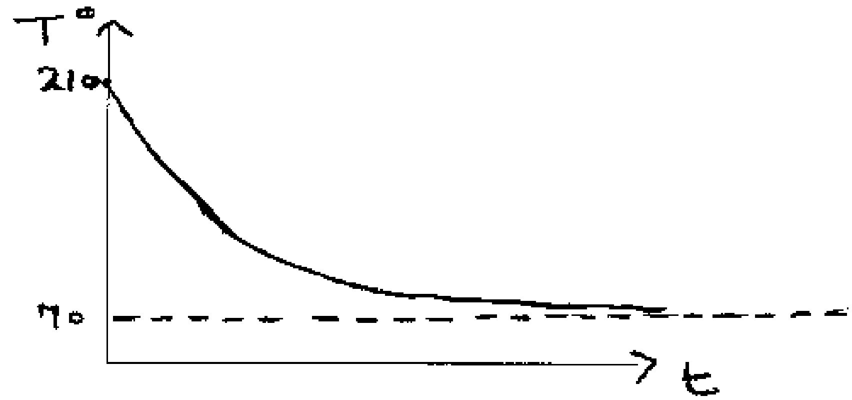
Distance covered by Speedy Racer in final three seconds is

$$.5(100) + .5(118) + .5(132) + .5(142) + .5(150) + .5(154) + .5(156) = 476.$$

Since she covers 476 feet in the final 3 seconds and $476 > 380$ she wins the race.

8. Marybeth pours boiling water into a cup to make tea. The water's temperature is 210 degrees Fahrenheit at the moment ($t = 0$) when she pours the tea. The temperature in her office is 70 degrees Fahrenheit.

(a) (4 points) On the axes provided below sketch a graph of the temperature T of the water as a function of the time after Marybeth finishes pouring. Be sure to include and label any asymptotes, maxima and minima, intercepts, etc.



(b) (4 points) The temperature T of the water, t minutes after it is poured is given by a formula of the form:

$$T = A - Be^{-0.5t}$$

Find the values of the constants A and B.

$t = 0 \Rightarrow 210 = A + B$

$t \to \infty \Rightarrow 70 = A$

Hence $B = 210 - 70 = 140$

$T = 70 + 140e^{-0.5t}$

(c) (4 points) Using the formula you obtained in part (b), determine the rate at which T is changing 2 minutes after the water has been poured. Your answer should be in the form of a complete sentence with units. (Round off your answer to 2 decimal places.)

$\frac{dT}{dt} = 140(-0.5)e^{-0.5t}$

$t = 2 \Rightarrow \frac{dT}{dt} = -70e^{-1} = -25.7515$

The temperature is decreasing at a rate of 25.75 degrees per minute.

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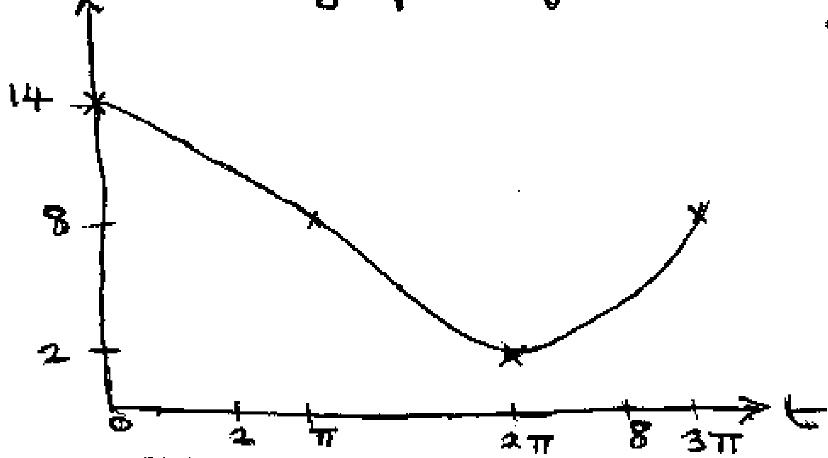
9. This question has parts (a), (b), and (c), on two pages.

Country Joe and the Fish are having a Fourth of July picnic on Half Moon Bay. The picnic lasts from 2:00 pm to 8:00 pm. The height H, in feet, of the water in Half Moon Bay t hours after noon on the Fourth of July, is given by the formula:

$$H = 6\cos\left(\frac{t}{2}\right) + 8$$

(a) (5 points) At what time **during the picnic** is the water highest? How high was the water at that time? Give your answers in a complete sentence. (Round off the height of the water to 2 decimal places and the time to the nearest minute.)

Graph of H looks like:



H is highest at $t = 2$

$H(2) = 6\cos(1) + 8$
$= 6(.54) + 8$
$= 11.24$

time = 2 p.m.
Height = 11.24 ft.

(b) (5 points) At what time **during the picnic** is the water level rising fastest and how fast is it rising at that time? Give your answers in a complete sentence. (Round off the height of the water to 2 decimal places and the time to the nearest minute.)

$$\frac{dH}{dt} = -6\sin\left(\frac{t}{2}\right)\frac{d}{dt}\left(\frac{t}{2}\right)$$
$$= -3\sin\left(\frac{t}{2}\right).$$

From graph see water rising fastest when $t = 8$

at $t = 8$, $\frac{dH}{dt} = -3\sin\left(\frac{8}{2}\right) =$

$-3\sin 4 = 3\sin(4 - \pi) = 2.27.$

Water level rises fastest at 8 p.m. It is rising at a rate of 2.27 feet per hour.

(c) (5 points) Estimate the average height of the water **between 2:00 pm and 5:00 pm**, i.e. during the first half of the picnic. Give your answer in the form of a complete sentence. Your estimate should be within .1 of the exact answer.

Average height between 2 and 5 is

$$\frac{1}{5-2}\int_2^5 H(t)\,dt =$$

$$\frac{1}{3}\int_2^5 \left[6\cos\left(\frac{t}{2}\right)+8\right] dt =$$

$$\frac{1}{3}\left[12\sin\left(\frac{t}{2}\right)+8t\right]_2^5 =$$

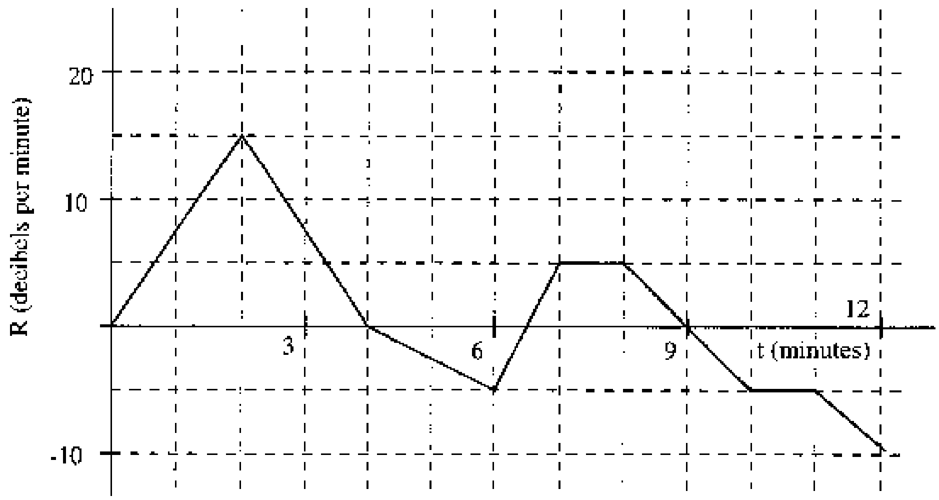
$$4\left[\sin\left(\frac{5}{2}\right)-\sin\left(\frac{3}{2}\right)\right]+8$$

$$= 6.404$$

Average height between 2 p.m. and 5 p.m. is 6.40 feet.

10. This question has parts (a) and (b) on two pages.

Suppose that at 10:00 pm Iggy and the Stooges begin playing their hit song "Raw Power." They begin playing at a volume of 90 decibels and the song takes 12 minutes to play (due to extended drum and guitar solos). The graph below represents the rate of change R of their volume, in decibels per minute, t minutes after 10:00 pm.



(a) (6 points) What is their volume at 10:06 pm (6 minutes after they begin playing "Raw Power")? Give your answer in a complete sentence.

volume at 10.06 p.m.

$= 90 + \int_0^6 R(t)\,dt$

$\int_0^4 R(t)\,dt =$ area of triangle

$= \frac{1}{2}$ base $\times$ height $= \frac{1}{2}(4)(15) = 30$

$\int_4^6 R(t)dt = -$ area of triangle

$= -\frac{1}{2}$ base $\times$ height $= -\frac{1}{2}(2)(5) = -5.$

$\int_0^6 R(t)\,dt = \int_0^4 R(t)\,dt + \int_4^6 R(t)\,dt = 30 - 5 = 25$

volume at 10.06 p.m. is $90 + 25 =$ 115 decibels.

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(b) (9 points) If their volume exceeds 120 decibels for more than 3 straight minutes, then listeners in the front row will sustain permanent ear damage. Did listeners in the front row sustain permanent ear damage during the song "Raw Power"? Explain your answer fully in complete sentences.

The volume first reaches 120 decibels at $t = 4$ but then decreases until $t = 6$, when it is 115. It decreases further until $t = 6\frac{1}{2}$ minutes but then increases again until $t = 9$ minutes. Observe that at $t = 7$ minutes the volume is again 115.

Have $t = 7 \Rightarrow Vol = 115$

$t = 8 \Rightarrow Vol = 115 + 5 = 120$

$t = 9 \Rightarrow Vol = 120 + 2.5 = 122.5$

$t = 10 \Rightarrow Vol = 122.5 - 2.5 = 120$

$t > 10 \Rightarrow Vol < 120.$

Volume is larger than 120 for only 2 minutes. Hence there is no permanent ear damage to listeners in the front row.

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ESSAY QUESTION

EXPLAIN your answers using complete sentences. Use graphs (labeled carefully and neatly) and formulas in your explanations.

11. (14 points) Ian and Becky are constructing a wooden box with a square bottom to hold their large collection of Lego blocks. Suppose that they use oak to make the top and bottom and pine to make the sides. Oak costs 6 dollars per square foot and pine costs 4 dollars per square foot. Their parents have given them 36 dollars to purchase wood for the box. What should the dimensions of the box be in order to maximize its volume? Explain your answer fully!

Let x be length of the side of the square bottom and h be the height. We wish to maximise the volume $V = x^2 h$.

cost of material = surface area of box times cost of material used.

area of top and bottom $= 2x^2$.
cost $= 6(2x^2) = 12x^2$.

area of 4 sides $= 4xh$
cost $= 4(4xh) = 16xh$.

Total cost $= 12x^2 + 16xh = 36$

$$\Rightarrow h = \frac{36 - 12x^2}{16x} = \frac{9}{4x} - \frac{3}{4}x$$

$$\text{Volume } V = x^2 h = x^2\left[\frac{9}{4x} - \frac{3}{4}x\right] = \frac{9x}{4} - \frac{3}{4}x^3.$$

$$\frac{dV}{dx} = \frac{9}{4} - \frac{9}{4}x^2 = \frac{9}{4}(1 - x^2).$$

(This page intentionally left blank.)

$$\frac{dV}{dx} < 0 \quad \text{if} \quad x < -1 \text{ or } x > 1$$

$$\frac{dV}{dx} > 0 \quad \text{if} \quad -1 < x < 1.$$

Hence V has a maximum in the region $0 < x < \infty$ at $x = 1$.

Hence $x = 1 \Rightarrow h = \frac{9}{4} - \frac{3}{4} = \frac{6}{4} = \frac{3}{2}$

Dimensions of the box should be $1 \times 1 \times \frac{3}{2}$ in feet.