Math 115 Calculus Exam II
March 25 1998

Department of Mathematics
University of Michigan

Name: ______________________ Instructor: ______________________

Signature: ______________________ Section: ______________________

General instructions: This test consists of 11 questions on 12 pages (including this cover sheet and a blank final page), totalling 100 points. When the exam begins, please count all of the pages of the exam, make sure none of them are missing, and write your name on each page.

Please read the instructions on each individual problem carefully, and indicate answers as directed. Show all your work! On questions 6–11 you can only be given credit for your answers if you show how you got them. If you found an answer using your calculator, briefly indicate what you did. If you are basing your reasoning on a graph from your calculator, sketch the graph. Write legibly. Use units wherever appropriate.

Problem	Points	Score
1	12	
2	6	
3	5	
4	8	
5	5	
6	8	
7	7	
8	12	
9	14	
10	8	
11	15	
Total	100	

NO PARTIAL CREDIT SECTION. (Problems 1–5.) No explanation necessary; no need to show work.

1. (12 points) The graph below describes the **derivative** $g'(x)$ of $g(x)$ (**NOT** $g(x)$ itself). At which of the marked values of x

since $g(x)$ decreases if $x < D$ i.e. $g'(x) < 0$, $x < D$ and increases if $x > D$.

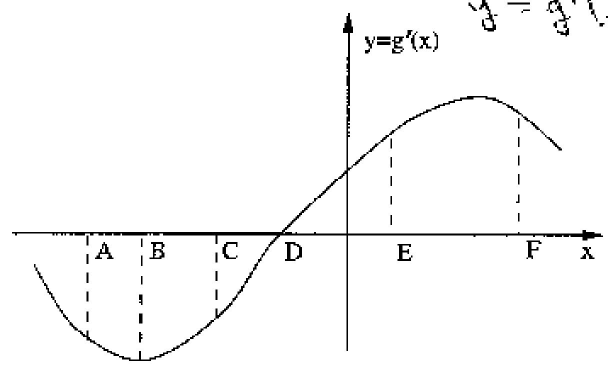
(a) is $g(x)$ least? $\boxed{D}$

(b) is $g(x)$ concave down and increasing? $\boxed{F}$

increasing $\Rightarrow$ $g'(x) > 0$
concave down $\Rightarrow$ $g'(x)$ decrease

(c) does $g(x)$ have an inflection point? $\boxed{B}$

inflection point $\Rightarrow$ $g''(x) = 0$
$\Rightarrow$ slope of tangent to $y = g'(x)$ is zero.



2. (6 points) The cost of buying a share of stock in King enterprises is rising faster each year. Suppose that $C(t)$ is the cost, in dollars, of buying a share of stock in King enterprises, t years after 1990. Circle each of the quantities below which must be positive:

$C(t)$ $\qquad$ $C'(t)$ $\qquad$ $C''(t)$

Cost always positive

Cost rises each year

Cost rises <u>faster</u> each year

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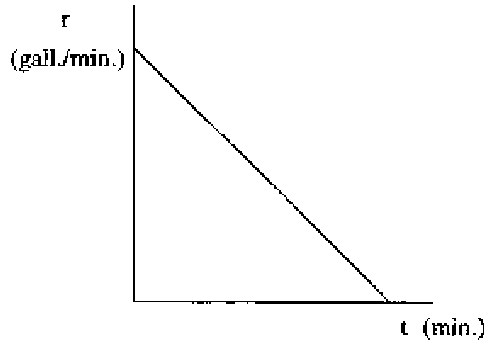
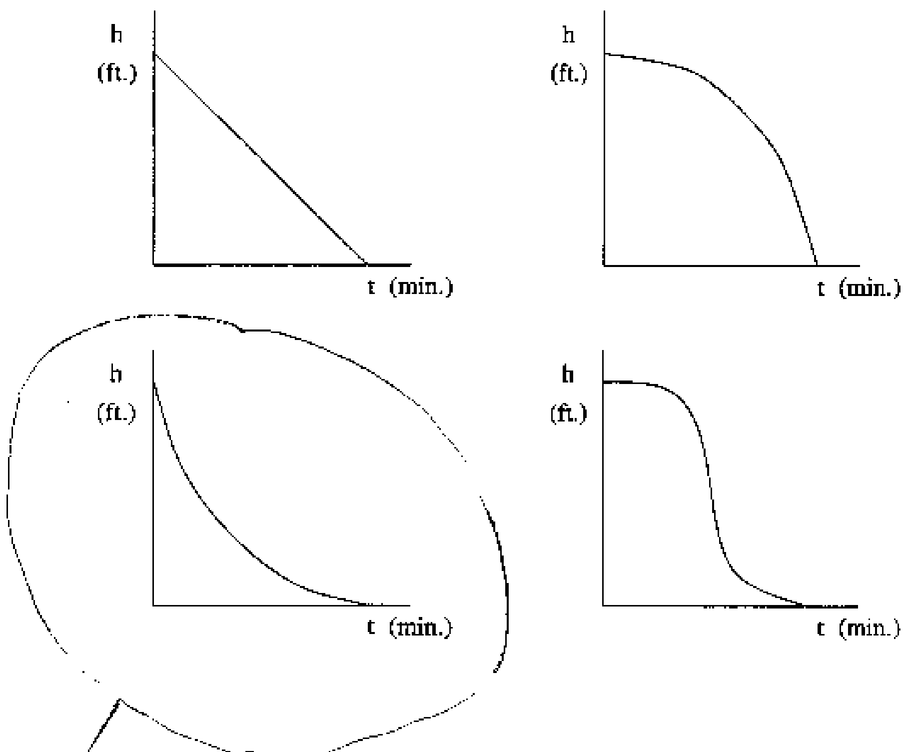
3. (5 points) A cubical water tank is drained completely through an opening in its bottom. The graph below shows the **rate** $r(t)$, in gallons per minute, at which the water flows out of the tank.



Circle the graph below which could be the graph of the height $h(t)$, in feet, of the water level in the tank during the process.



$h'(t) = -c\,r(t)$ where $c > 0$ is constant.

Hence $h''(t) = -c\,r'(t) > 0 \implies$ graph of h is convex.

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4. Let f and g be functions described by the following table of data.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	1	−5	−2
2	2	3	7	−3
3	4	2	5	4
4	1	4	9	6

(a) (4 points) Let $h(x) = g(f(x))$. Compute $h'(3)$.

$$h'(3) = g'(f(3))f'(3)$$

$$= g'(4)f'(3) = 6 * 5 = 30.$$

Answer: $\boxed{30}$

(b) (4 points) Let $m(x) = 3f(x) + g(x)$. Compute $m'(2)$.

$$m'(2) = 3f'(2) + g'(2)$$
$$= 3(7) + (-3)$$
$$= 21 - 3 = 18$$

Answer: $\boxed{18}$

5. (5 points) Write the equation of a function $f(x)$ such that $f(0) = 1$ and $f(x) = f'(x)$ for all values of x.

$$f(x) = a f'(x) \Rightarrow f(x) = A e^{a \cdot x}$$
$$f(0) = 1 \Rightarrow A = 1.$$

Answer: $\boxed{f(x) = e^x}$

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LIMITED PARTIAL CREDIT SECTION. (Problems 6–7.) In this section, you need not explain your answers, but you must show your work. Very limited partial credit may be possible.

6. (8 points) Find the equation of the tangent line to the graph of $f(x) = e^{x^2-4}$ at the point where $x = 2$. (Your answer must be **exact** and **not** just approximate.)

$y = e^{x^2-4} = e^{z}, \quad z = x^2 - 4.$

$f'(x) = \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = e^{z}(2x) = 2xe^{x^2-4}$

$f'(2) = 4e^{4-4} = 4e^{0} = 4.$

Equation of tangent line is

$y - f(2) = f'(2)[x-2] \Rightarrow$

$y - 1 = 4[x-2] \Rightarrow y = 4x - 7$

Answer: $y = 4x - 7$

7. (7 points) Consider the equation:

$$x^2 + xy^2 = 17 + 2y$$

Find the value of $\frac{dy}{dx}$ at the point where $x = 3$ and $y = 2$.

$\frac{d}{dx}[x^2 + xy^2] = \frac{d}{dx}[17 + 2y] \Rightarrow$

$2x + \frac{d}{dx}(xy^2) = 0 + 2\frac{dy}{dx} \Rightarrow$

$2x + y^2 + x\frac{d}{dx}(y^2) = 2\frac{dy}{dx} \Rightarrow$

$2x + y^2 + 2xy\frac{dy}{dx} = 2\frac{dy}{dx}$. Put $x = 3, y = 2 \Rightarrow$

$2(3) + 2^2 + 2(3)(2)\frac{dy}{dx} = 2\frac{dy}{dx} \Rightarrow 10\frac{dy}{dx} = -10 \Rightarrow$

$\frac{dy}{dx} = -1$

Answer: $\frac{dy}{dx} = -1$

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SHORT ANSWER SECTION. (Problems 8–10.) Limited partial credit may be possible, and a few sentences of explanation may be required. You must show your work to receive credit.

8. George Bedard is due onstage at the Ark at 8:15 pm. He leaves Ashley's at 8:00 pm and jogs to the Ark, arriving at 8:06 pm. Suppose that $g(t)$ is the total distance, in yards, that George Bedard covers in the first t minutes of his jog.

t	0	1	2	3	4	5	6
$g(t)$	0	110	250	420	590	760	900

(a) (6 points) What was George Bedard's average velocity during his trip from Ashley's to the Ark? Give your answer in a complete sentence with units.

average velocity = $\frac{\text{distance gone}}{\text{time taken}}$

$= \frac{900 \text{ yards}}{6 \text{ minutes}} = 150$ yards / minute.

His average velocity was 150 yards per minute.

(b) (6 points) Estimate George Bedard's instantaneous velocity at 8:04pm. Give your answer in a complete sentence with units.

At 8.04 p.m. George has gone 590 yards.

Average velocity between 8.04 & 8.05

$= 760 - 590 = 170$ yards / minute

Average velocity between 8.03 & 8.04

$= 590 - 420 = 170$ yards / minute.

Hence our best estimate for instantaneous velocity at 8.04 p.m is 170 yards per minute.

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9. This question has parts (a), (b) and (c) on two pages.

Ian is on a hunger strike to protest his parent's refusal to let him watch the cartoon *Park City*. Suppose that he weighs $w(t)$ pounds t days after he begins his hunger strike.

(a) (4 points) **EXPLAIN** what $w'(4) = -2$ means in practical terms, using no more than **TWO** sentences.

It means that his rate of weight loss is 2 pounds per day after 4 days.

(b) (4 points) Using only the facts that $w(4) = 91$ and $w'(4) = -2$, estimate how much Ian weighed 3 days after he began his hunger strike. Give your answer in the form of a complete sentence.

$$w'(4) = \lim_{h \to 0} \frac{w(4+h) - w(4)}{h}.$$ It was

$$w'(4) \simeq \frac{w(4+h) - w(4)}{h}$$ for small h.

Put $h = -1 \Rightarrow$

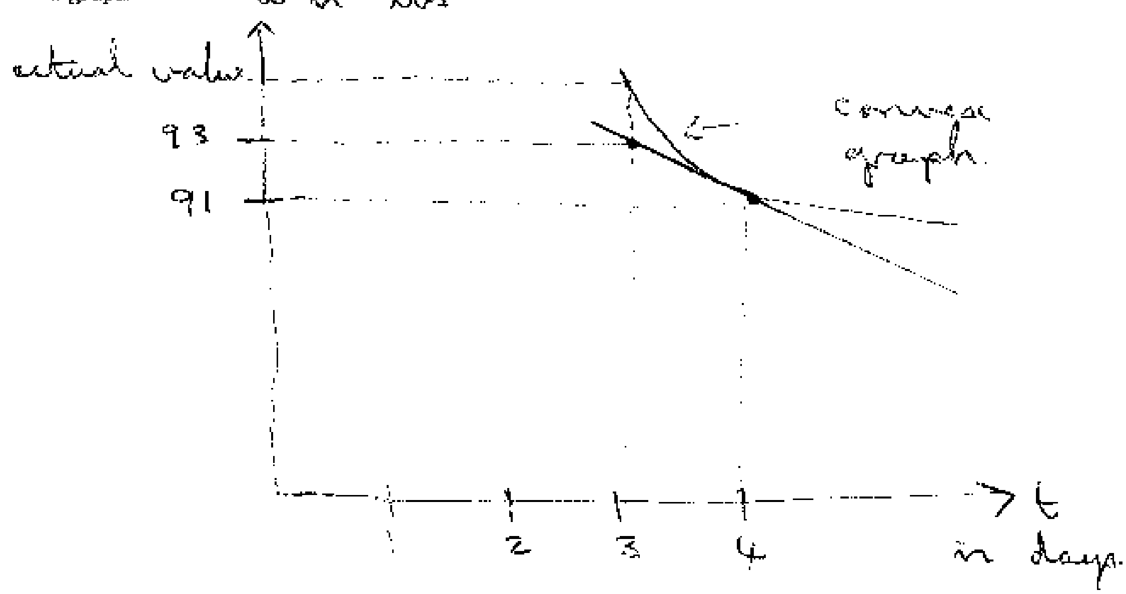
$$w'(4) \simeq \frac{w(3) - w(4)}{-1}$$

$$-2 \simeq 91 - w(3) \Rightarrow$$

$$w(3) \simeq 91 + 2 = 93 \text{ lbs.}$$

Ian's weight was approximately 93 lbs on the 3rd day of the strike.

(c) (6 points) If, in addition to the facts in parts (a) and (b), you are told that $w''(4) = 0.5$, do you expect that your estimate in part (b) is more or less than Ian's actual weight 3 days after he began his hunger strike? Explain your answer in one or two complete sentences. Illustrate your answer with a graph.



Since $w''(4) > 0$ the graph of w is convex close to $t = 4$ days. Our estimate in (b) is the tangent line approximation at $t = 4$. A convex curve lies above the tangent line. Hence we think our estimate in (b) is too low.

10. (8 points) Suppose that Becky has spilled her soda and a circular pool of soda is forming beneath her chair. Three seconds after she spilled her soda, the circular pool of soda has a radius of 12 centimeters and its area is growing at a rate of 60 square centimeters per second. How fast is the radius of the pool of soda changing 3 seconds after Becky spilled her soda?

NO EXPLANATION NECESSARY, but show all your work, and give your answer in a complete sentence. Round off your final answer to 2 decimal places.

r = radius of pool in centimeters
A = area of pool in square centimeters.

$A = \pi r^2$.

$t = 3$ seconds $\Rightarrow$ $r = 12$ cms.

$$\frac{dA}{dt} = 60$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$t=3 \Rightarrow \frac{dA}{dt} = 60, \ r = 12 \Rightarrow$$

$$60 = 2\pi(12)\frac{dr}{dt} \Rightarrow$$

$$\frac{dr}{dt} = \frac{5}{2\pi} \text{ cms / sec.}$$

$$= .7958 \text{ cms / sec.}$$

The radius of the pool is increasing at a rate of .80 cms / sec 3 seconds after the spill.

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ESSAY QUESTION

EXPLAIN your answers using complete sentences. Use graphs (labeled carefully and neatly) and formulas in your explanations.

11. This question has parts (a) and (b) on two pages.

Suppose that $f(x) = x^3 - cx^2 + 7$ for some positive constant $c > 0$.

(a) (10 points) Find all the critical points of $f(x)$ and decide whether each critical point is a local maximum, a local minimum or neither. Explain your answer fully.

$$f'(x) = 3x^2 - 2cx$$
$$= x[3x - 2c]$$

critical points when $f'(x) = 0$

ie. $x = 0$ and $x = \frac{2c}{3}$.

$x > \frac{2c}{3} \Rightarrow f'(x) > 0$

$0 < x < \frac{2c}{3} \Rightarrow f'(x) < 0$

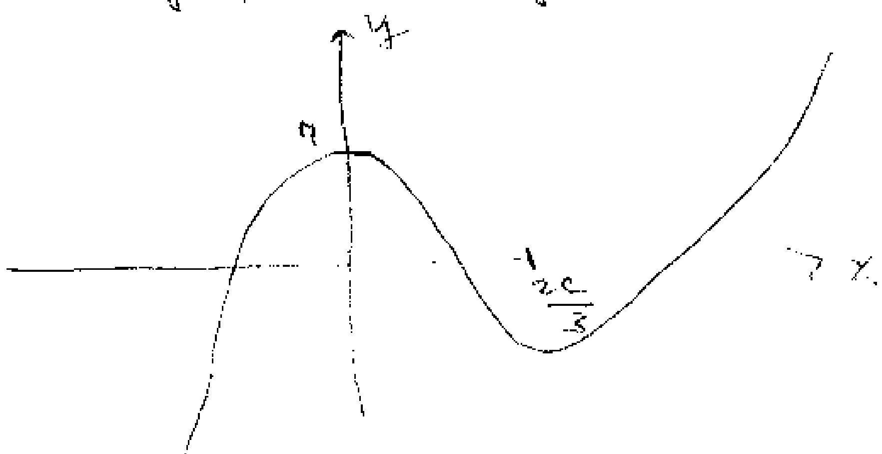
$x < 0 \Rightarrow f'(x) > 0$.

Hence $x = \frac{2c}{3}$ is a local minimum

and $x = 0$ is a local maximum.

(b) (5 points) Find all the global maxima and global minima of $f(x)$, if any. Explain your answer fully.

graph of $y = f(x)$ looks like.



Hence $\lim_{x \to +\infty} f(x) = +\infty$

and $\lim_{x \to -\infty} f(x) = -\infty$.

We conclude there are no global maxima or minima.