1.1. RELATIONS

Defining relations as sets of ordered pairs

Any relation naturally leads to pairing. Let us illustrate this with an example.

1.1.1. EXAMPLE.

John is 23, Bob is 25, Elizabeth is 21 and Sylvia is 27 years old. Let us consider the following relation: the first person is related to the second person if the first person is older than the second person. It leads to the following set of pairs of related persons:

$$\{(Sylvia, John), (Sylvia, Bob), (Sylvia, Elizabeth),$$
$$(Bob, John), (Bob, Elizabeth), (John, Elizabeth)\}.$$

Abstractly, we define relation to be a set of ordered pairs. In this setting, we consider the first element of the ordered pair to be related to the second element of the ordered pair.

1.1.2. DEFINITION.

If X and Y are sets then any set of ordered pairs (x, y), where x is an element of X and y is an element of Y, is called a relation.

1.1.3. EXAMPLE.

Let us consider the following set of ordered pairs of numbers

$$R = \{(2, 5), (4, 3), (6, 1), (2, 7)\}.$$

According to the definition the above set gives us a relation between the numbers $1, 2, 3, 4, 5, 6, 7$. It is not hard to see which numbers are related and which are not. For example, 6 is related to 1 because the ordered pair $(6, 1)$ belongs to the set R. The number 5 is not related to 2 because the pair $(5, 2)$ is not an element of the set R (because of the order the pair $(5, 2)$ is not the same as the pair $(2, 5)$).

1.1.4. EXERCISES I.

1. Exercise.

   Which of the following set does not define a relation?

   A. $\{(2, 1), (2, 2), (2, 3), (2, 5)\}$

B. $\{(2,5),(3,4),5,(1,2)\}$

C. $\{(2,2)\}$

D. $\{(2,3),(3,2)\}$

Got to answer **??**

2. Exercise.

Which of the following sets defines a relation which relates 2 to 3?

A. $\{(2,1),(2,2),(2,5),(3,4)\}$

B. $\{(3,2),(3,3),(7,4),(5,2)\}$

C. $\{(1,3),(2,3),(6,1),(4,3)\}$

D. $\{1,2,4,5\}$

Go to answer **??**

1.1.5. ANSWERS I.

1. Answer to Exercise 1 is "B"

   The set $\{(2,5),(3,4),5,(1,2)\}$ does not define a relation because it has the number 5 as an element. Of course, the single number does not form an ordered pair of numbers. Go back **??**

2. Answer to Exercise 2 is "C"

   The set $\{(1,3),(2,3),(6,1),(4,3)\}$ contains the ordered pair $(2,3)$. Go back **??**

Describing relations through words

The relation between parents and children could described by simply saying "Mary is the mother of John" and "Ted is the father of John". Understanding that the relation is being a parent, we could also say "Mary is related to John" and "Ted is related to John". In general, the above relation could be described by saying "person A is related to person B if A is a parent of B". In describing relation through words we describe the rule according to which the ordered pairs of elements appear. Remember that we need to provide sufficient information to decide whether the given elements are related or not related.

1.1.6. EXAMPLE.

The description "number k is related to number m if the difference $k - m$ is positive" provides the all necessary information. We can easily decide if 5 is related to 7 or if 0.5 related to 0.35. We have $5 - 7 = -2$ which is negative so 5 is not related to 7. In the similar way we can decide that 0.5 is related to 0.35 because $0.5 - 0.35 = 0.15$ is positive. Let us also point out that the above relation could be also described by "a number k is related to the number m if $k > m$".

1.1.7. EXERCISES II.

1. Exercise. The set

$$\{(2, 4), (4, 2), (2, 6), (6, 2), (4, 6), (6, 4), (2, 2), (4, 4), (6, 6)\}$$

defines a relation between the numbers $1, 2, 3, 4, 5, 6, 7$. Describe the above relation in words. Go to answer **??**

2. Exercise. If we say "number x is related to number y if x and y", is it a description of a relation? Explain your answer. Go to answer **??**

3. Exercise. Let the relation be defined by "number a is related to number b if $ab > 0$". Is the number 7 related to the number 0.5? Is 0.325 related to $-1, 250$? How can we describe the above relation through words differently? Go to answer **??**

4. Exercise. If the relation is "number k is related to number m if the difference $k - m$ is even" is 7 related to 4 ? is 7 related to 5 ? How can we describe the above relation through words differently? Go to answer **??**

1.1.8. ANSWERS II.

1. Answer to Exercises 1 is "C".

   Let a and b be numbers from the set $\{1, 2, 3, 4, 5, 6, 7\}$. The number a is related to the number b if a and b are even. Go back **??**

2. Answer to Exercise 2 is "No".

   Saying "number x is related to number y if x and y" does not give us a description of a relation because there is no information provided that allows us to decide whether or not 2 is related 3. Go back **??**

3. Answer to Exercise 3 is "Yes".

   "Number a is related to number b if $ab > 0$" describes a relation. The number 7 is related to the number 0.5 because $7(0.5) = 3.5 > 0$. The number 0.325 is not related to $-1,250$ because $0.325(-1.250) = -0.40625 < 0$. The above relation could be described through words as follows "number a is related to number b if either a and b are both positive or a and b are both negative. Go back **??**

4. Answer to Exercise 4 is "Yes".

   "Number k is related to number m if the difference $k-m$ is even" describes a relation. The number 7 is not related to 4 because $7 - 4 = 3$ is odd but 7 is related to 5 because $7 - 5 = 2$ is even. The above relation could be described through words as follows "number k is related to number m if either k and m are both even or k and m are both odd. Go back **??**

Describing relations through tables

A relationship between two variables X and Y can be also described in the form of a table.

1.1.9. EXAMPLE. The table below describes a relation between two variables X and Y in which the change by 1 in X always corresponds to the change by 4 in Y.

X	0	1	2	3	4	5	6
Y	20	24	28	32	36	40	44

While exploring the world around us we want to know how certain quantities are related. The best way to find this out is gathering "real world" data, putting them in the worksheet format, and performing the data analysis. The typical worksheet format is a table.

1.1.10. EXAMPLE. Statistical Office of the United Nations provides the following information about the world population (in millions) from 1980 through 1992.

Year	1980	1985	1986	1987	1988	1989	1990	1991	1992
Population	4453	4850	4936	5024	5112	5202	5294	5384	5478

Describing relations through equations and inequalities

If the number of ordered pairs in the set defining a relation is very large or even infinite then the best way to describe the relation is through using an equation or an inequality.

1.1.11. EXAMPLE. Let us consider a relation described by the equation $x^4 + y^2 = 10$. The number 1 is related to the number 3 because after the substitution $x = 1$ and $y = 3$ we obtain a true equality $1^4 + 3^2 = 10$. The number 3 is not related to the number 1 because after the substitution $x = 3$ and $y = 1$ to the left side of the equation we obtain $3^4 + 1^2 = 82$ which give us a false equality $82 = 10$. In the set notation the above relation could be described as $\{(x,y)|x^4 + y^2 = 10\}$.

1.1.12. EXAMPLE. Another example of a relation is an inequality $x > 3y-1$. The number 3 is related to 1 because the substitution $x = 3$ and $y = 1$ gives a true inequality $3 > 2$. The number 5 is not related to 3 because the substitution $x = 5$ and $y = 3$ gives a false inequality $5 > 8$. In the set notation the above relation could be described as $\{(x,y)|x > 3y - 1\}$