1.4. Functions as special relations

Defining functions as better relations

Let us compare the relations

$$\{(2,3),(2,5),(1,5),(3,4)\}$$

and

$$\{(3,2),(2,5),(1,5),(3,4)\}.$$

The first relation contains the pairs $(2,3)$ and $(2,5)$ which means that 2 is related to 3 and 5. In the second relation 2 is related to 5 only. Interpreting the first element of an ordered pair as an input and the second element as an output we can say that in the first relation the input 2 gives two outputs 3 and 5 but in the second relation the input 2 gives the output 5 only. Obviously, in real world applications a relation in which each input gives uniquely determined output provides a better and more useful description of a given situation.

1.4.1. DEFINITION.

A relation R between the elements of a set X and the elements of a set Y is called a function if each x from the domain of R is related to one and only element y from the range of R.

The example below explains why functions are more useful than relations.

1.4.2. EXAMPLE.

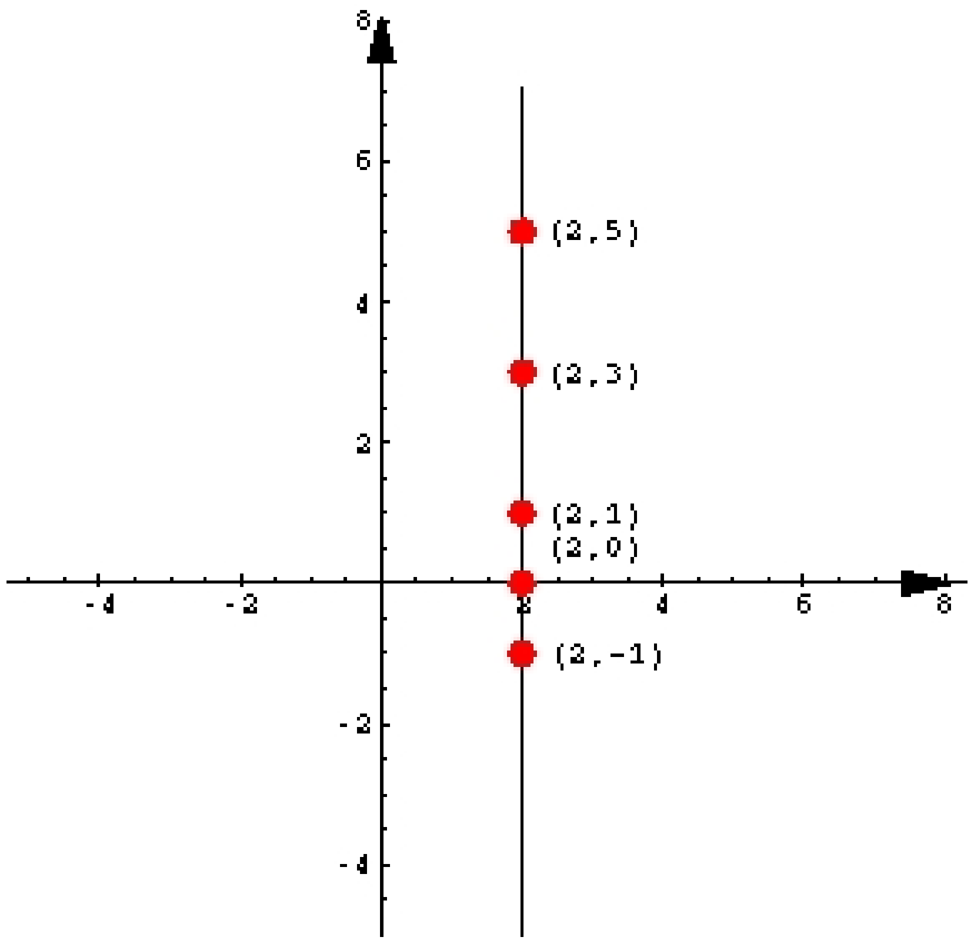
A manager of a chain of grocery stores is researching how the amount of tomatoes sold at given time depends on the price. Gathering data, the manager is looking for a relation to describe the number of pounds of tomatoes sold at different prices. It turned out that one day there were 2000 pounds of tomatoes sold at the price \$.69 per pound and the other day the stores sold 3000 pounds of tomatoes at same price. The manager cannot decide which of the two numbers represent better the amount of tomatoes sold at the price \$.69. The manager would like the relation obtained by gathering data to be a function primarily because in addition to being a relation, a function has the property of giving a single value for the amount of tomatoes sold at each price.

1.4.3. EXAMPLE.

Is the relation given by the equation $x - y^2 = 0$ a function? The answer is no because the above equation relates $x = 4$ to $y = 2$ and to $y = -2$. Even more, if x is a positive number then x is related to $y = \sqrt{x}$ and to $y = -\sqrt{x}$.

Applying the vertical line test

A vertical line consists of all points which have x-coordinates the same. The figure below shows the vertical line consisting of all points having x-coordinate equal to 2. This vertical line is the graph of the equation $x = 2$ (in words it says that 2 is related to every real number.)



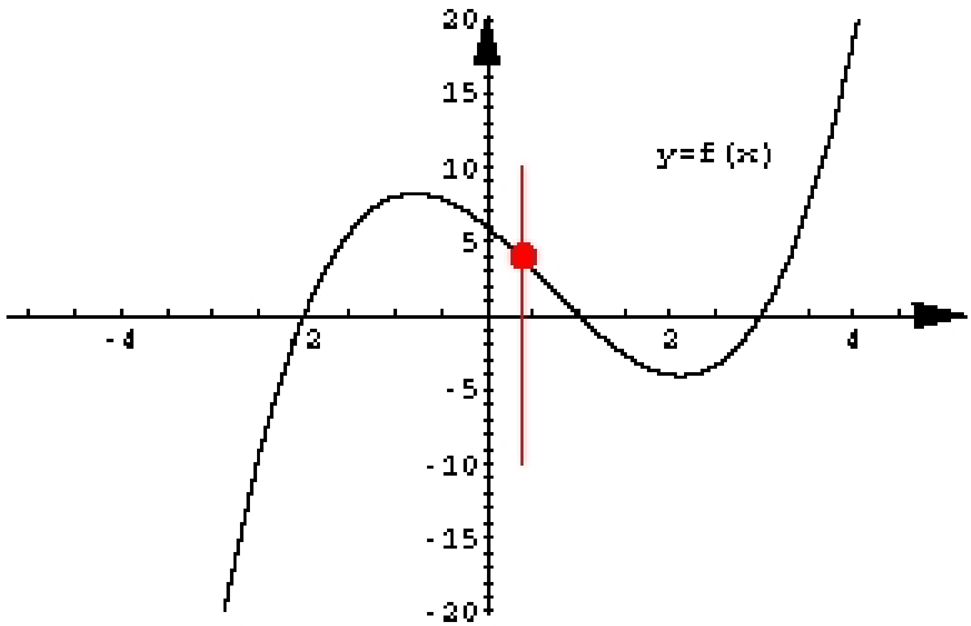
By the definition of a function, given value of x is related to at most one value of y. It follows, that a vertical line can intersect the graph of a function at most once.

1.4.4. VERTICAL LINE TEST.

A set of points in a coordinate plane is the graph of a function if and only if no vertical line intersects the graph at more than one point.

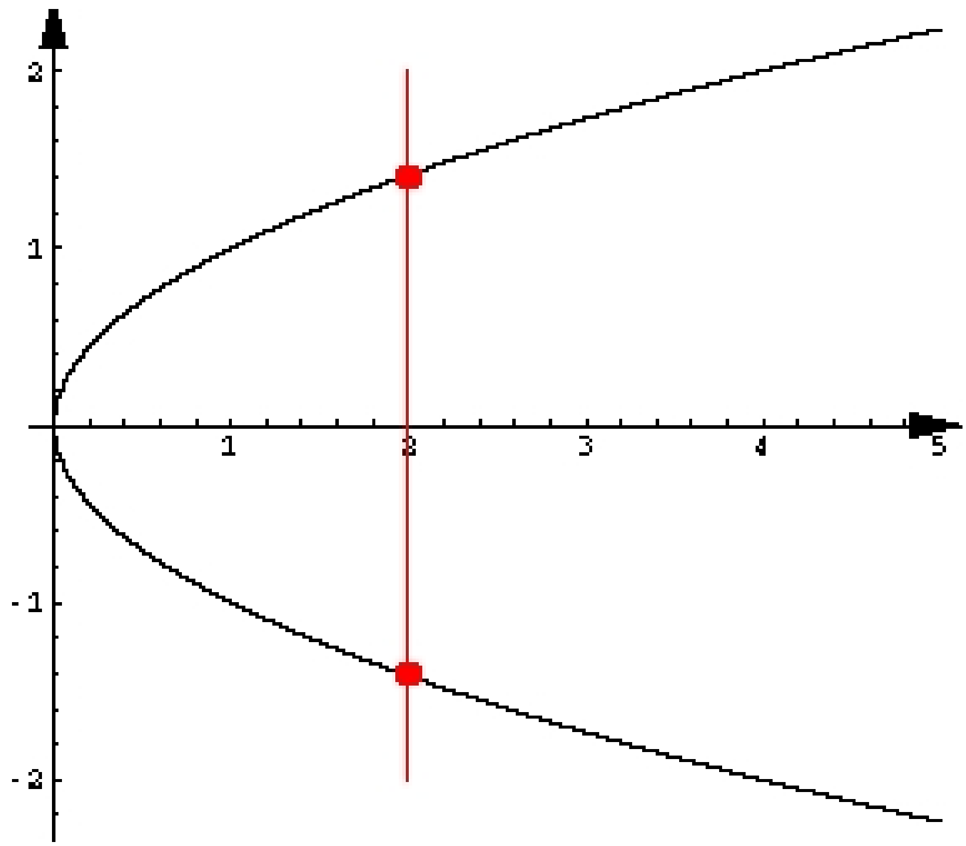
1.4.5. EXAMPLE.

The graph below represents a function.



1.4.6. EXAMPLE.

The graph below does not represent a function.



Understanding function notation

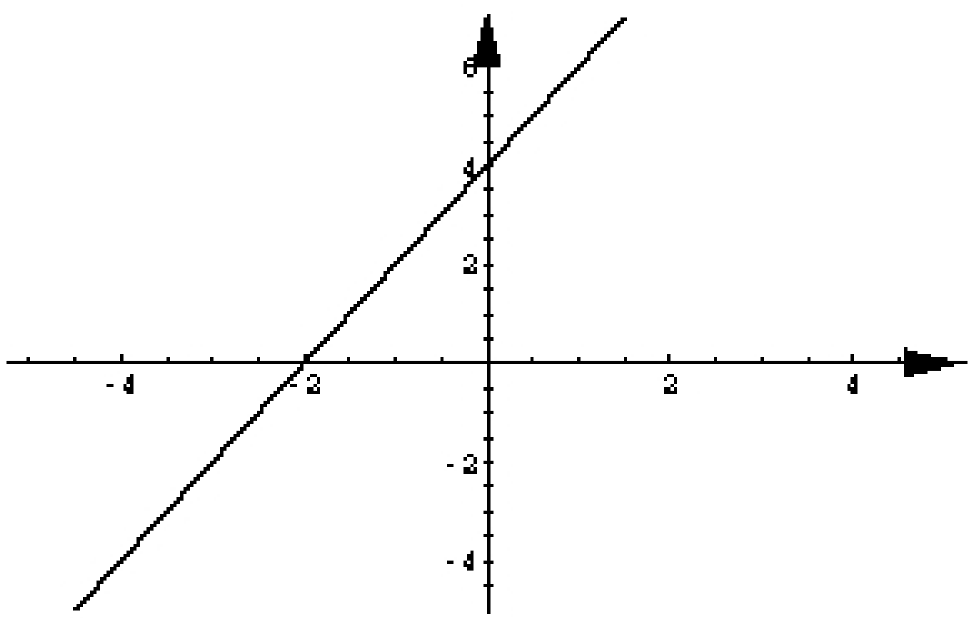
Some functions are defined by equation of the form $y = f(x)$, where the letter f denotes the formula to which the value of x has to be substituted in order to obtain the value of y. For instance, the f in the equation $f(x) = 3x + 5$ tells you that in order to get the value of y we need to multiply the value of x by 3 and add 5. So the equation $f(x) = 3x + 5$ for the input $x = 2$ gives us the output 11. Briefly, we write $f(2) = 11$ saying that value of the function f at 2 is 11. The variable x (input) is called the independent variable and y (output) is called the dependent variable.

Graphing functions

The graph of a function $y = f(x)$ is the set of all points (x, y) such that the y-coordinate is the value of the function f at the given value of the x-coordinate. It means that the graph of the function f is the set of all points $(x, f(x))$, where x is taken from the domain of the function f.

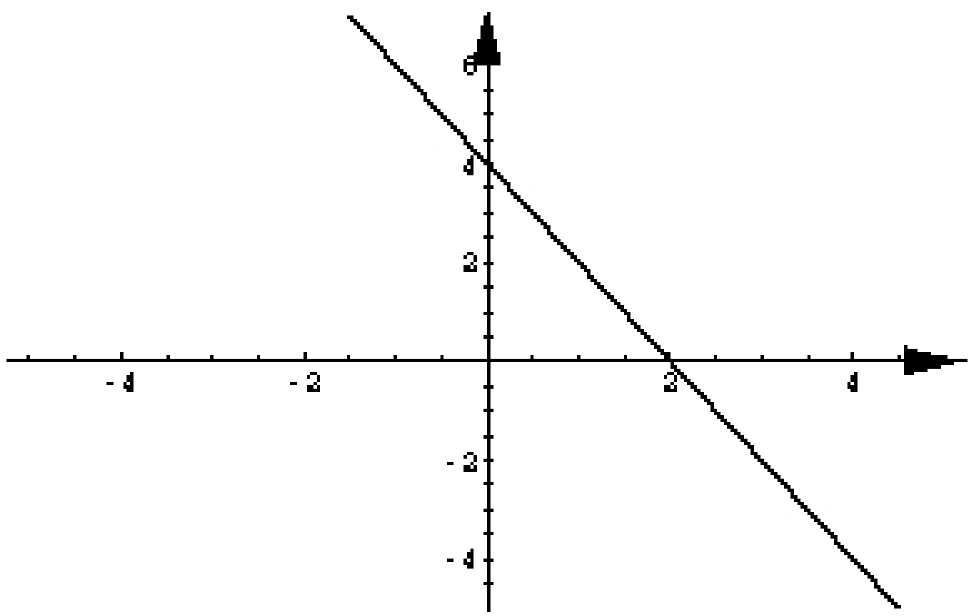
A function is *increasing* over an interval (a, b) if for greater input from (a, b) we obtain a greater output.

1.4.7. EXAMPLE. The figure below shows a graph of a function which is increasing.



A function is *decreasing* over an interval (a, b) if for greater input it gives smaller output.

1.4.8. EXAMPLE. The figure below shows a graph of a function which is decreasing



1.4.9. EXERCISES.

1. Exercise. Which of the following relations is a function?

   A. $x = y^4 + 1$

   B. $\{(2,3),(1,3),(5,3),(17,3)\}$

   C. $y < x$

   D. $\{(2,3),(1,2),(5,2),(2,17)\}$

   Go to answer 1

2. Exercise. Which of the following points lie on the same vertical line?

   A. $\{(x,y)|y = 3\}$

   B. $(2,3)$, $(2,27)$, and $(1,2)$

   C. $(5,5)$, $(5,4)$, and $(5,7)$

   D. $\{(x,y)|x^2 = 4\}$

   Go to answer 2

3. Exercise. Which of the following points lie on the same horizontal line?

   A. $\{(x,y)|y^2 = 9\}$

   B. $(2,3)$, $(2,27)$, and $(2,5)$

   C. $(5,5)$, $(4,5)$, and $(7,5)$

   D. $\{(x,y)|x = 4\}$

   Go to answer 3

4. Exercise. Which of the following equations describe a function?

   A. $y^2 = 6x^4 - 5x^2 + 1$

B. $y^2 = 3x^4 + 1$

C. $|y| = |x|$

D. $y = \sqrt{x^2 + 1}$

Go to answer 4

5. Exercise. Which of the following functions is decreasing in $(-\infty, \infty)$?

A. $y = -5x$

B. $y = \frac{1}{5}x$

C. $y = x^2$

D. $y - 10x = 2$

Go to answer 5

1.4.10. ANSWERS

1. Answer to Exercise 1 is "B".

   Go back 1

2. Answer to Exercise 2 is "C".

   Go back 2

3. Answer to Exercise 3 is "C".

   Go back 3

4. Answer to Exercise 4 is "D".

   Go back 4

5. Answer to Exercise 5 is "A".

   Go back 5