1.5. Transformations of graphs

Translating graphs vertically

Let c be a positive number.

Translation upward. The vertical translation c units upward shifts given point onto the point which is located in the same vertical line c units above the given point. Analytically, the vertical translation c units upward shifts a point (x, y) onto the point $(x, y + c)$, briefly

$$(x, y) \mapsto (x, y + c).$$

If the graph of a function $y = f(x)$ is translated vertically c units upward then each point $(x, f(x))$ is shifted onto the point $(x, f(x) + c)$ i.e.

$$(x, f(x)) \mapsto (x, f(x) + c).$$

It means that the translated graph represents the function defined by the equation $y = f(x) + c$, schematically

$$y = f(x) \mapsto y = f(x) + c.$$

Translation downward. The vertical translation c units downward shifts given point onto the point which is located in the same vertical line c units below the given point. Since all points located in the same vertical line have x-coordinates equal the vertical translations change only the y-coordinates of points. Analytically, the vertical translation c units downward shifts a point (x, y) onto the point $(x, y - c)$ i.e.

$$(x, y) \mapsto (x, y - c).$$

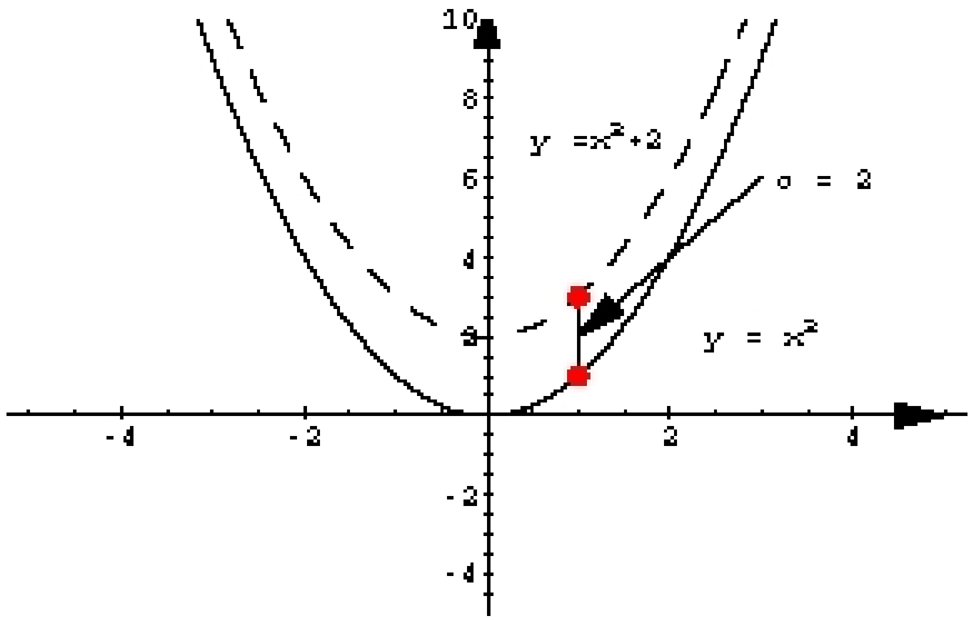
If the graph of a function $y = f(x)$ is translated vertically c units downward then each point $(x, f(x))$ is shifted onto the point $(x, f(x) - c)$ i.e.

$$(x, f(x)) \mapsto (x, f(x) - c).$$

It means that the translated graph represents the function defined by the equation $y = f(x) - c$

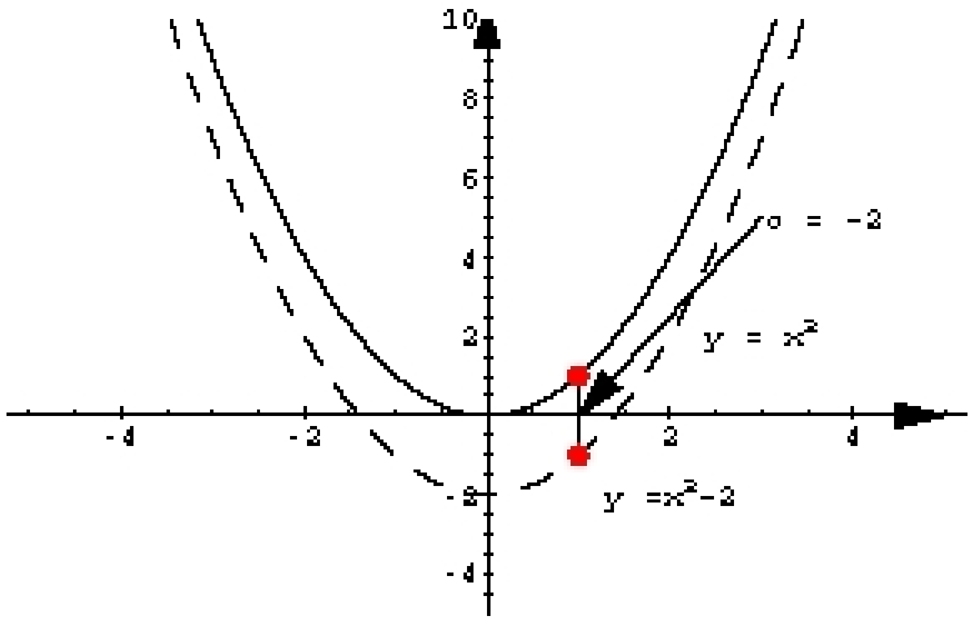
$$y = f(x) \mapsto y = f(x) - c.$$

1.5.1. EXAMPLE.



The figure above illustrates the vertical translation 2 units upward of the graph of the function $y = x^2$.

1.5.2. EXAMPLE.



The figure below illustrates the vertical translation 2 units downward of the graph of the function $y = x^2$.

Translating graphs horizontally

Let c be a positive number.

Translation to the left. The horizontal translation c units to the left shifts given point onto the point which is located in the same horizontal line c units to the left of the given point. The horizontal translation c units to the right shifts given point onto the point which is located in the same horizontal line c units to the right of the given point. Since all points located in the same horizontal line have y-coordinates equal horizontal translations change only the x-coordinates of points. Analytically, the horizontal translation c units to the left shifts a point (x, y) onto the point $(x - c, y)$

$$(x, y) \mapsto (x - c, y).$$

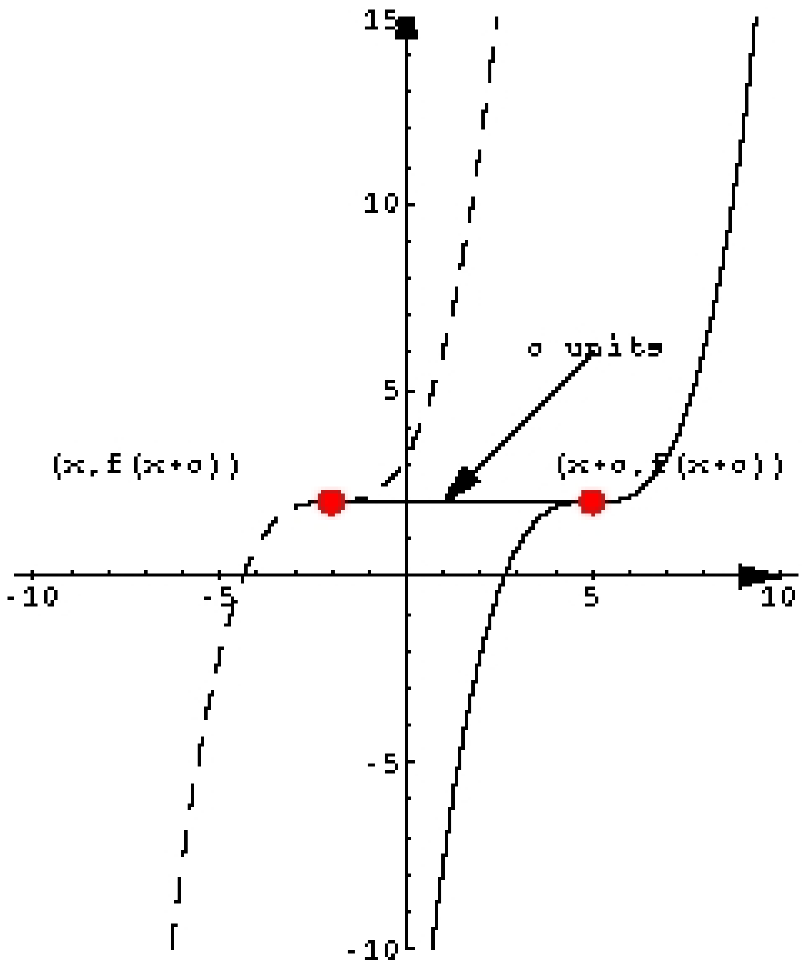
If the graph of a function $y = f(x)$ is translated horizontally c units to the left then each point $(x, f(x))$ is shifted onto the point $(x - c, f(x))$

$$(x, f(x)) \mapsto (x - c, f(x)).$$

In order to find the equation of the function represented by the translated graph we need to analyze how the y-coordinate of the point $(x - c, f(x))$ is expressed in terms of the x-coordinate $x - c$. We let $\overline{x} = x - c$. Then $x = \overline{x} + c$ and $f(x) = f(\overline{x} + c)$. Hence the equation that we are looking for will have the form $y = f(\overline{x} + c)$. By renaming $\overline{x}$ by x again we obtain $y = f(x + c)$.

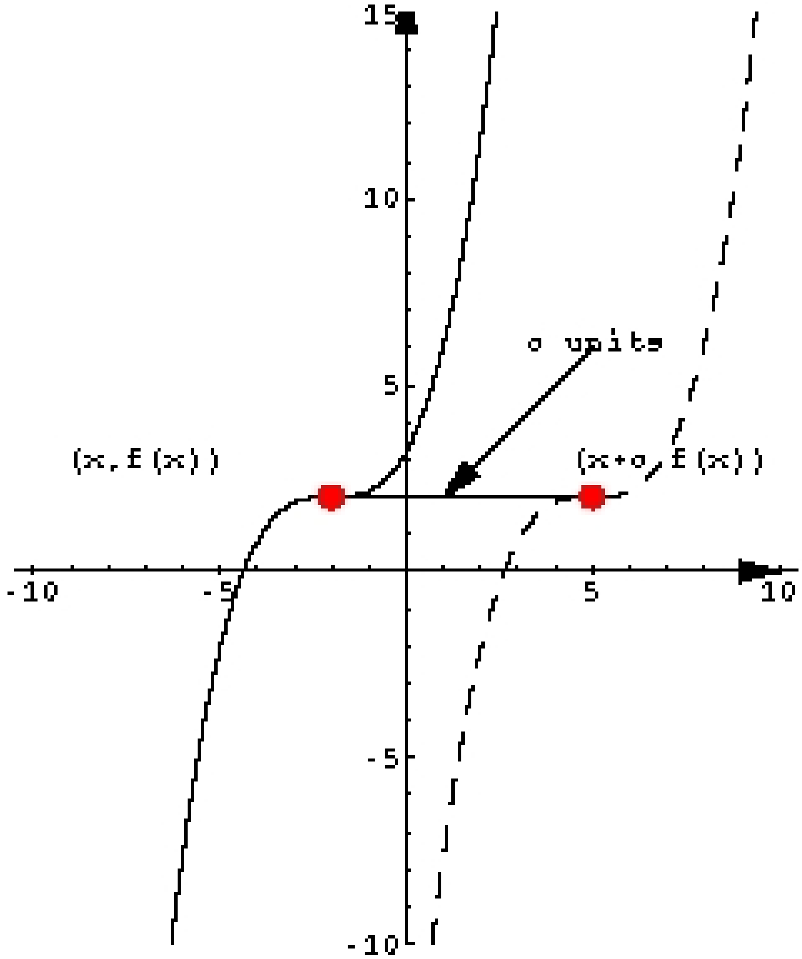
Translation to the right. The horizontal translation c units to the right shifts a point (x, y) onto the point $(x + c, y)$. If the graph of a function $y = f(x)$ is translated horizontally c units to the right then each point $(x, f(x))$ is shifted onto the point $(x + c, f(x))$ with the x-coordinate equal to $x + c$. We have $(x + c, f(x)) = (x + c, f((x + c) - c))$. In order to identify the function represented by the translated graph we need to analyze how the y-coordinate of the point $(x + c, f((x + c) - c))$ is expressed in terms of the x-coordinate $x + c$. Since the x-coordinate is $x + c$ and the y-coordinate is $f((x + c) - c))$ we see that in order to get the y-coordinate we need to apply the formula f not to the x-coordinate but to the x-coordinate decreased by c. It means that the translated graph represents the function defined by the equation $y = f(x - c)$.

1.5.3. EXAMPLE.



The figure above illustrates the horizontal translation c units to the left.

1.5.4. EXAMPLE.

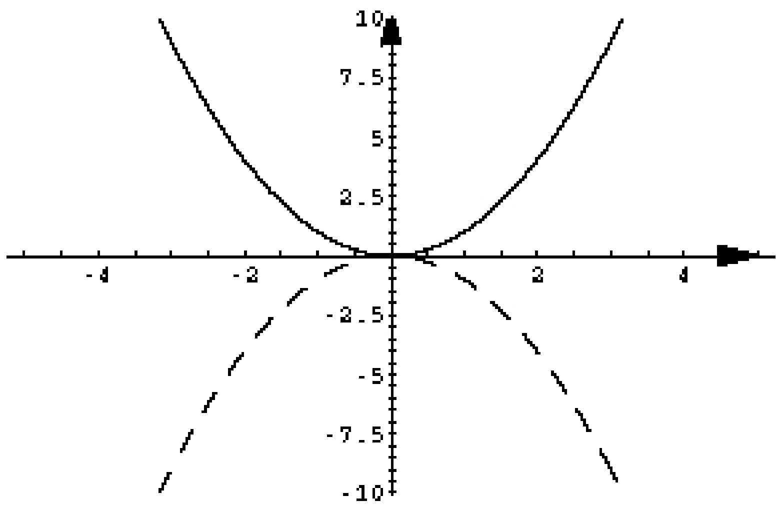


The figure above illustrates the horizontal translation c units to the right.

Reflecting graphs in x-axis

The reflection in the x-axis maps given point onto the point which is located in the same vertical line on the opposite side of the x-axis but in the same distance from the x-axis as the given point. Since all points located in the same vertical line have x-coordinates equal the reflection in the x-axis changes only the y-coordinates of points. The reflection in the x-axis maps a point (x, y) onto the point $(x, -y)$. If the graph of a function $y = f(x)$ is reflected in the x-axis then each point $(x, f(x))$ is mapped onto the point $(x, -f(x))$ with the y-coordinate equal to $-f(x)$. It means that the reflected graph represents the function defined by the equation $y = -f(x)$.

1.5.5. EXAMPLE.

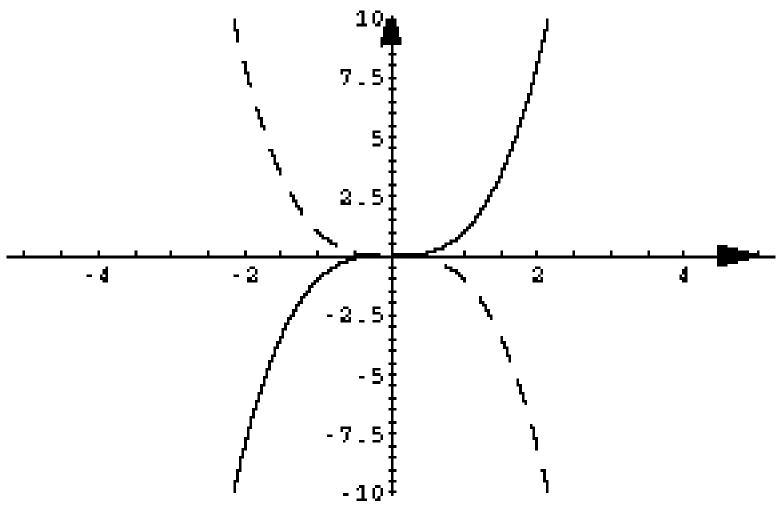


The figure above illustrates the reflection of the graph of the function $y = x^2$ in x-axis.

Reflecting graphs in y-axis

The reflection in the y-axis maps given point onto the point which is located in the same horizontal line on the opposite side of the y-axis but in the same distance from the y-axis as the given point. Since all points located in the same horizontal line have y-coordinates equal the reflection in the y-axis changes only the x-coordinates of points. The reflection in the y-axis maps a point (x, y) onto the point $(-x, y)$. If the graph of a function $y = f(x)$ is reflected in the x-axis then each point $(x, f(x))$ is mapped onto the point $(-x, f(x))$ with the y-coordinate equal to $f(x) = f(-(-x))$. It means that the reflected graph represents the function defined by the equation $y = f(-x)$.

1.5.6. EXAMPLE.

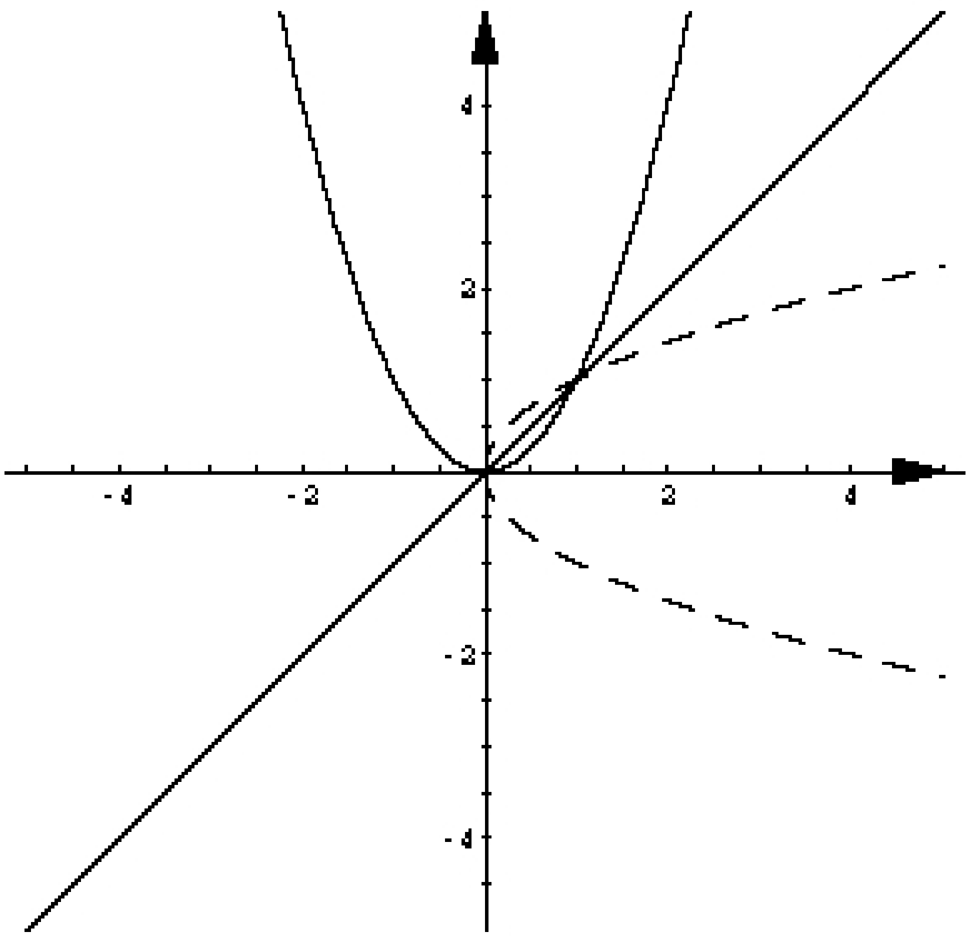


The figure above illustrates the reflection of the graph of the function $y = x^3$.

Reflecting graphs in the diagonal line $y = x$

The reflection in the diagonal line $y = x$ maps the given point onto the point which is located in the same line perpendicular to $y = x$ on the opposite side of the line $y = x$ but in the same distance from the line $y = x$ as the given point (mirror image in the line $y = x$). The reflection in the line $y = x$ interchanges x and y. Thus it maps a point (x, y) onto the point (y, x). If $y = f(x)$ is the equation describing a graph then $x = f(y)$ is the equation describing the reflected graph.

1.5.7. EXAMPLE.



The figure above illustrates the reflection in the line $y = x$ of the graph of the function $y = x^2$. The obtained graph is not a function but it can be described by the equation $x = y^2$.

Stretching and shrinking graphs

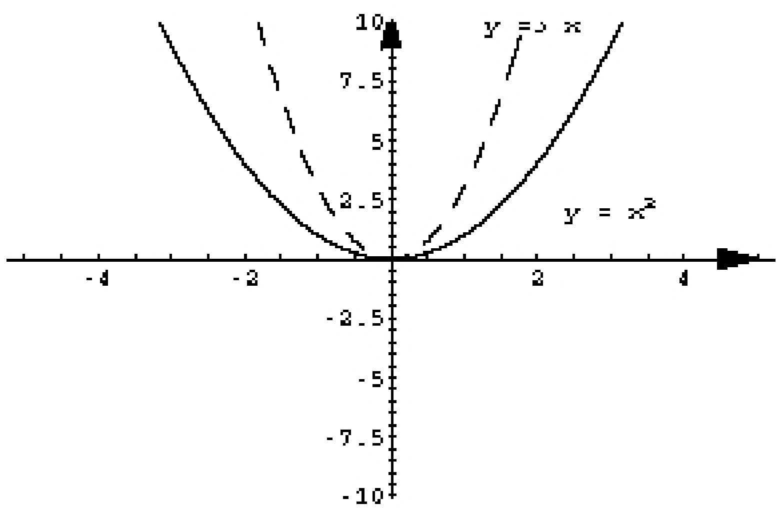
Translations and reflections do not change the basic shape of the original graph and are called rigid transformations. Obviously, stretching and shrinking cause a change in the shape of the original graph.

1.5.8. DEFINITION.

If $c > 1$ is a number then the transformation which maps a point (x, y) onto the point (x, cy) is called a vertical stretch. If $0 < c < 1$ is a number then the transformation which maps a point (x, y) onto the point (x, cy) is called a vertical shrink.

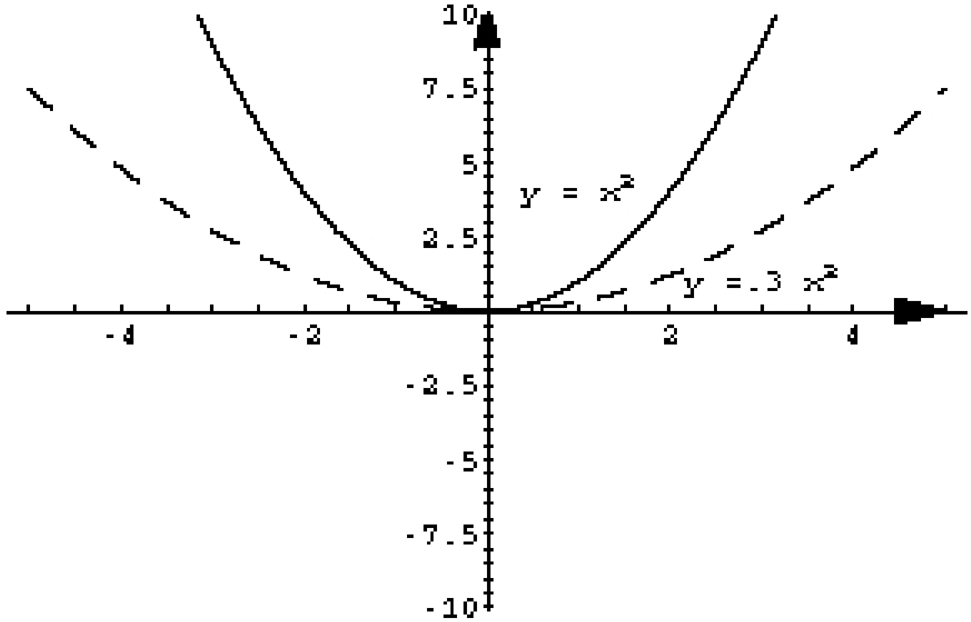
The vertical stretch or shrink maps points $(x, f(x))$ onto points $(x, cf(x))$ and transforms the graph of a function $y = f(x)$ onto the graph of the function $y = cf(x)$.

1.5.9. EXAMPLE.



The figure above shows the graphs of the functions $y = x^2$ and $y = 3x^2$

1.5.10. EXAMPLE.



The figure above shows the graphs of the functions $y = x^2$ and $y = 0.3x^2$