1.7. Operations on functions

Equality of functions

In order to define the operations on functions we need to understand what it means that two functions are equal. The equality of two functions does not mean the same as equality of two numbers (numbers have a fixed value but values of functions vary). Because each function is a relationship between x and y, the two relationships are the same if for every value of x (for every input) we get the same value of y (the same output).

1.7.1. EXAMPLE.

The functions $y = x$ and $y = x^2$ are not equal because for the input $x = 2$ the first one gives the output 2 and the second one gives the output 4.

1.7.2. EXAMPLE.

The functions $(x-1)(x+2)$ and $x^2 + x - 2$ are equal by the distribution law.

1.7.3. EXAMPLE.

The following is another example of two functions which are equal for positive values of x

$$|x| = \sqrt{x^2}$$

Algebraic operations on functions

If f and g are functions, then for all values of x for which both $f(x)$ and $g(x)$ exist,

the sum of f and g is defined by

$$(f+g)(x) = f(x) + g(x)$$

,

the difference of f and g is defined by

$$(f-g)(x) = f(x) - g(x)$$

,

the product of f and g is defined by

$$(fg)(x) = f(x)g(x)$$

,

the quotient of f and g is defined by

$(\frac{f}{g})(x)$, where $g(x) \neq 0$

1.7.4. EXAMPLE.

If $f(x) = 5x + 1$ and $g(x) = x^2 + 3$ then

the sum of f and g is

$$(f + g)(x) = (5x + 1) + (x^2 + 3) = x^2 + 5x + 4$$

,

the difference of f and g is

$$(f - g)(x) = (5x + 1) - (x^2 + 3) = -x^2 + 5x - 2$$

,

the product of f and g is

$$(fg)(x) = (5x + 1)(x^2 + 3) = 5x^3 + x^2 + 15x + 3$$

,

the quotient of f and g is

$$(\frac{f}{g})(x) = \frac{5x + 1}{x^2 + 3}$$

Composite functions

If f and g are functions, then the composition (composite function) of f and g is defined by

$$(g \circ f)(x) = g(f(x))$$

for all values of x in the domain of f such that $f(x)$ is in the domain of g. The mathematical symbol $g(f(x))$ means that, in order to find the y value of the composite function corresponding to given value of x, we start with finding the value $f(x)$ of the function f, and then substitute the obtained value $f(x)$ for x in the formula for the function g.

WARNING. The order of function in the composition is important. It is very common that

$$(f \circ g)(x)(g \circ f)(x)$$

.

1.7.5. EXAMPLE.

If $f(x) = 5x + 1$ and $g(x) = x^2 + 3$ then

the composition of f and g is

$(g \circ f)(x) = (5x + 1)2 + 3 = 25x^2 + 10x + 4$, and the composition of g and f is

$(f \circ g)(x) = 5(x^2 + 3) + 1 = 5x^2 + 16$.