

## 1.8. One-to-one functions

### *Defining one-to-one functions*

A function relates each value of the independent variable  $x$  (input) to the single value of the dependent variable  $y$  (output). It is possible that two or more different inputs give us the same output. In order to understand it better let us consider the function  $y = x^3 - 6x^2 + 11x - 6$ . By the simple substitution we see that the inputs  $x = 1$ ,  $x = 2$  and  $x = 3$  give us the same output  $y = 0$ . In this case three values of  $x$  are related to one value of  $y$  (it seems to be three-to-one instead one-to-one). Functions for which the same output does not repeat for different inputs are called one-to-one.

#### 1.8.1. DEFINITION.

A function  $y = f(x)$  is called an *one-to-one* function if for each  $y$  from the range of  $f$  there exists exactly one  $x$  in the domain of  $f$  which is related to  $y$ .

#### 1.8.2. EXAMPLE.

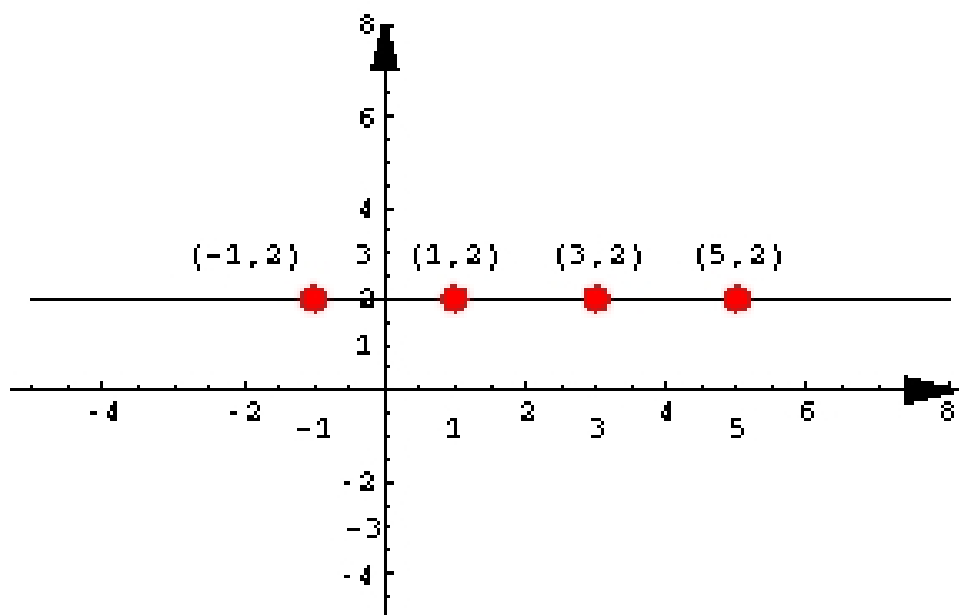
Let us compare the functions  $\{(2, 3), (4, 5), (1, 5), (3, 4)\}$  and  $\{(2, 3), (4, 2), (1, 5), (3, 4)\}$ . The first function contains the pairs  $(4, 5)$  and  $(1, 5)$  which means that the inputs 4 and 1 give the same output 5, so the first function is not one-to-one. There is no repetition of outputs of the second function which means that the function is one-to-one.

#### 1.8.3. EXAMPLE.

Let us compare the functions  $y = x^2$  and  $y = 3x + 1$ . Since the first function repeats the output  $y = 4$  for the inputs  $x = 2$  and  $x = -2$  ( $4 = 2^2$  and  $4 = (-2)^2$ ), the function is not one-to-one. There is no repetition of outputs of the second function which means that the function is one-to-one.

*The horizontal line test*

A *horizontal line* consists of all points which have  $y$ -coordinates equal to the same number. The figure below shows the horizontal line consisting of all points having  $y$ -coordinate equal to 2. This line could be described by the equation  $y = 2$  (in words the equations says that every value of  $x$  is related to 2.)



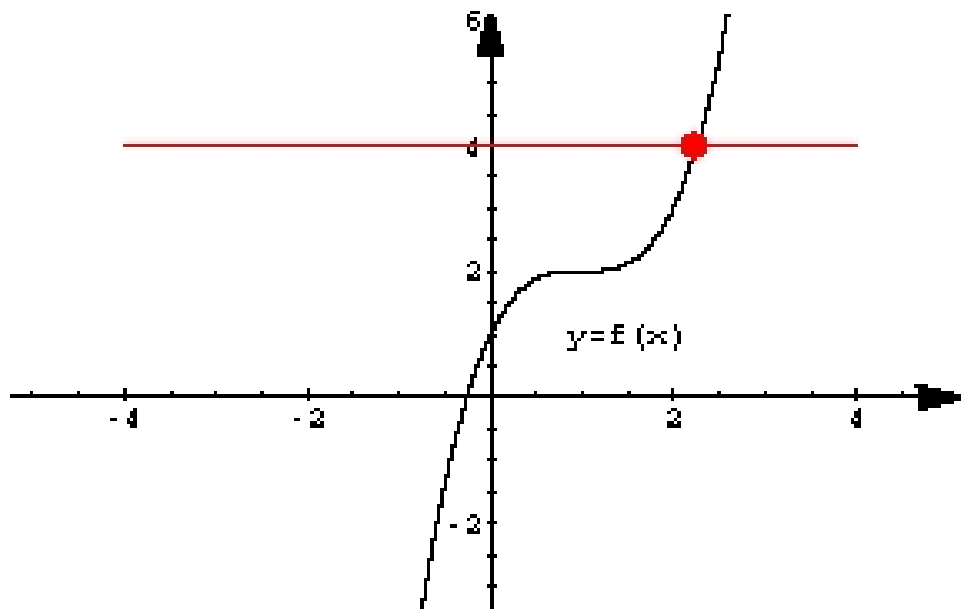
By the definition of an one-to-one function, at most one value of  $x$  is related to the given value of  $y$ . It follows, that a horizontal line can intersect the graph of a function at most once.

#### 1.8.4. HORIZONTAL LINE TEST.

The graph of a function in a coordinate plane is the graph of an one-to-one function if and only if no horizontal line intersects the graph at more than one point.

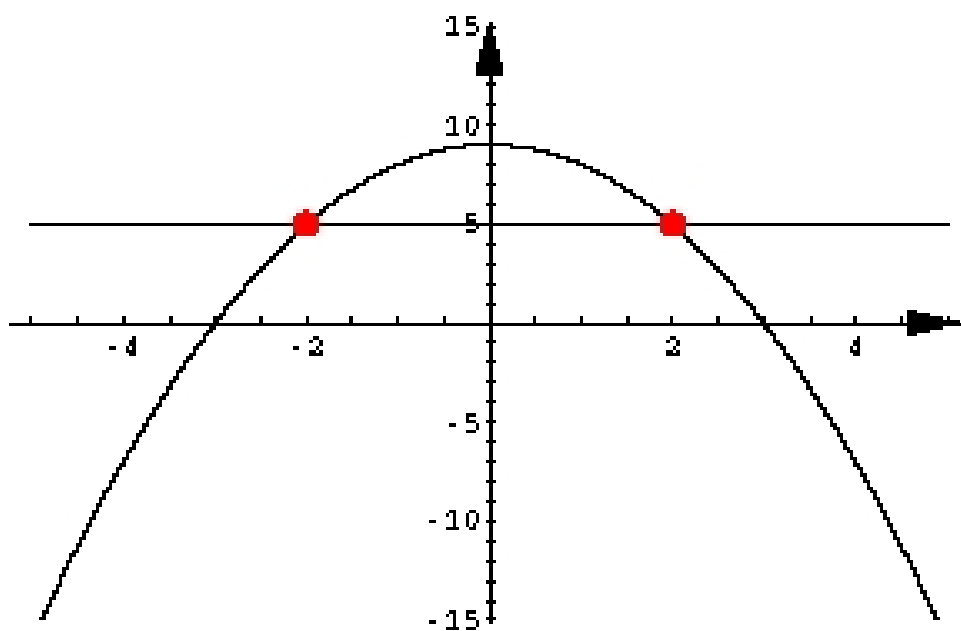
#### 1.8.5. EXAMPLE.

The graph below represents a one-to-one function.



1.8.6. EXAMPLE.

The figure below shows the graph of the function  $y = 9 - x^2$ . We can see that the graph does not represent a one-to-one function because it has two intersections with the horizontal line  $y = 9$ .



*Identifying one-to-one functions algebraically*

1.8.7. EXAMPLE.

We have checked that the function  $f(x) = 9 - x^2$  in the Example 1.8.6 is not one-to-one by applying the Horizontal Line Test. Now, we will show how to do it algebraically. First, we need to change the functional notation into an equation in  $x$  and  $y$ . The substitution  $y = f(x)$  give us  $y = 9 - x^2$ . Now, we solve the equation for the variable  $x$ .

$$y = 9 - x^2$$

$$0 = 9 - x^2 - y$$

$$x^2 = 9 - y$$

If  $y < 9$  then  $9 - y$  is positive and the equation above has two solution for  $x$ ,  $x = \sqrt{9 - y}$  or  $x = -\sqrt{9 - y}$ . For instance, for  $y = 5$  we obtain  $x = 2$  or  $x = -2$ . The inputs 2 and  $-2$  give the same output 5. It means that the function is not one-to-one.

1.8.8. EXERCISES.

1. Exercise. Which of the following functions is one-to-one?

A.  $y = 3$

B.  $\{(2, 3), (1, 2), (5, 2), (3, 17)\}$

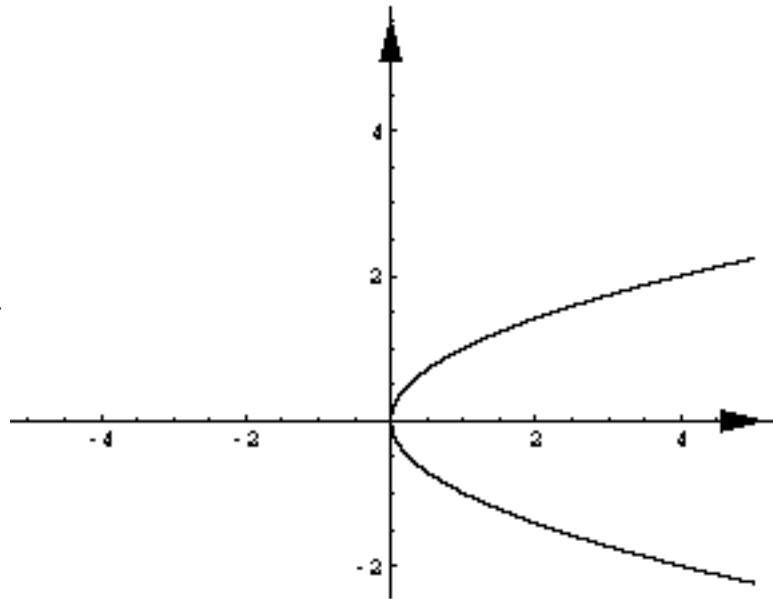
C.  $y = |x|$

D.  $\{(2, 3), (1, 2), (5, 1), (3, 17)\}$

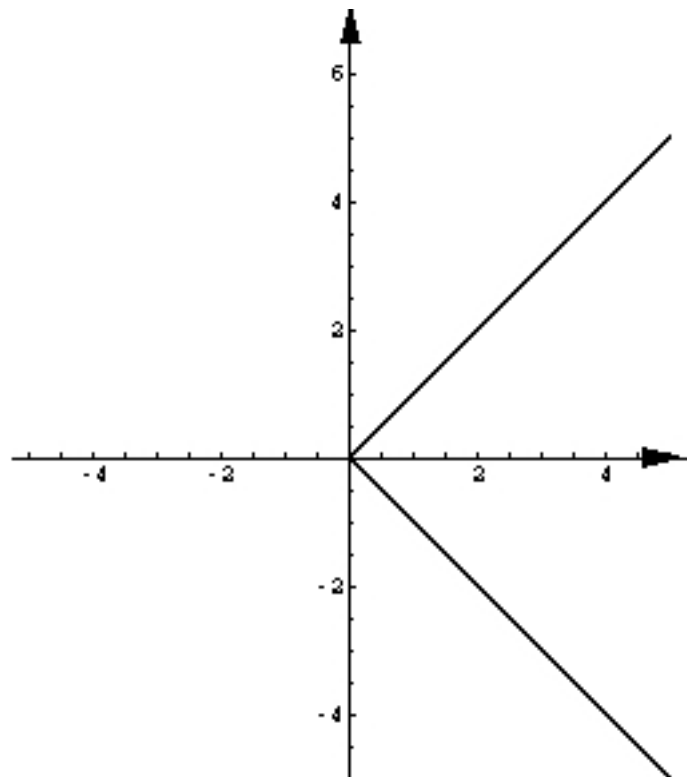
Go to answer 1

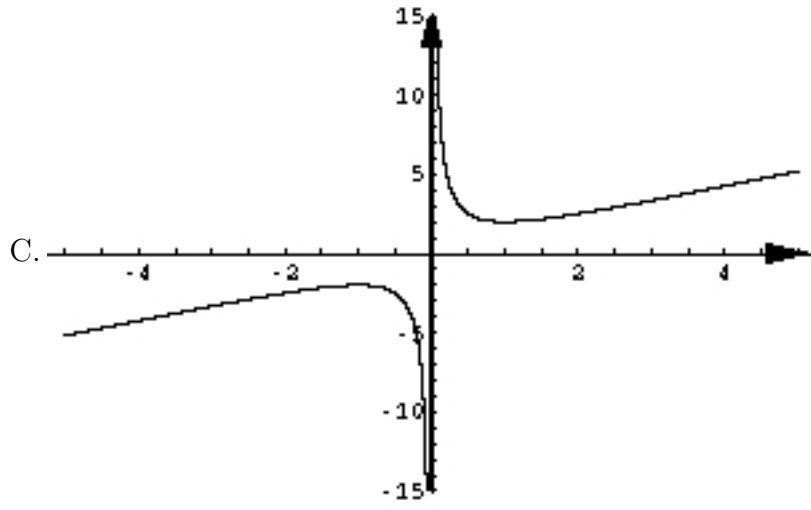
2. Exercise. Which of the following graphs represents a one-to-one function?

A.

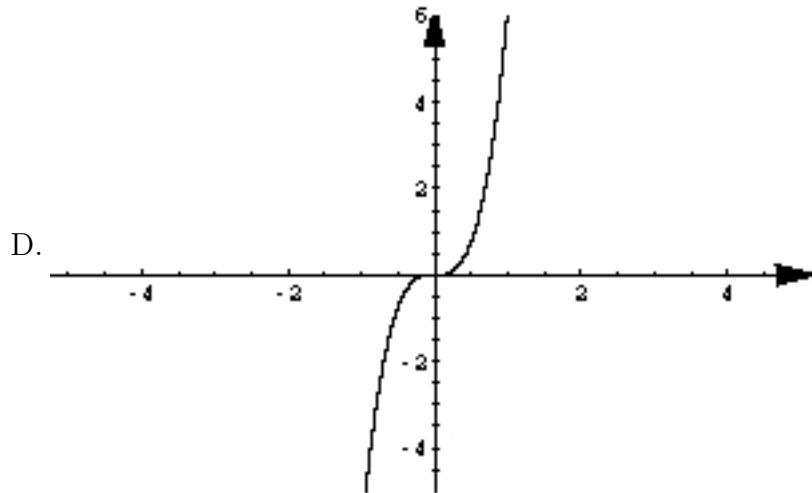


B.









Go to answer 2

3. Exercise. The function given by the equation  $y = x^2 - 2x + 1$  is not a one-to-one function because
- A. two is related to one so not one-to-one
  - B. there is one value of  $x$  related to two values of  $y$
  - C. if  $y = 4$  then the equation  $y = x^2 - 2x + 1$  has two solutions  $x = -1$  and  $x = 3$
  - D. one input gives two different outputs

Go to answer 3

1.8.9. ANSWERS.

1. Answer to Exercise 1 is "D".

Go back 1

2. Answer to Exercise 2 is "D".

Go back 2

3. Answer to Exercise 3 is "C".

Go back 3