

1.8. One-to-one functions

Defining one-to-one functions

A function relates each value of the independent variable x (input) to the single value of the dependent variable y (output). It is possible that two or more different inputs give us the same output. In order to understand it better let us consider the function $y = x^3 - 6x^2 + 11x - 6$. By the simple substitution we see that the inputs $x = 1$, $x = 2$ and $x = 3$ give us the same output $y = 0$. In this case three values of x are related to one value of y (it seems to be three-to-one instead one-to-one). Functions for which the same output does not repeat for different inputs are called one-to-one.

1.8.1. DEFINITION.

A function $y = f(x)$ is called an *one-to-one* function if for each y from the range of f there exists exactly one x in the domain of f which is related to y .

1.8.2. EXAMPLE.

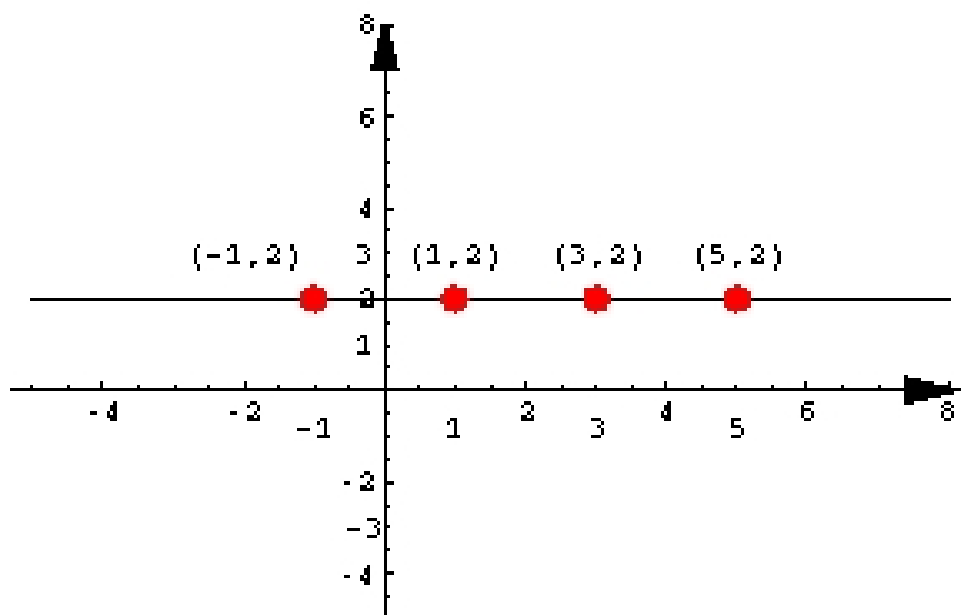
Let us compare the functions $\{(2, 3), (4, 5), (1, 5), (3, 4)\}$ and $\{(2, 3), (4, 2), (1, 5), (3, 4)\}$. The first function contains the pairs $(4, 5)$ and $(1, 5)$ which means that the inputs 4 and 1 give the same output 5, so the first function is not one-to-one. There is no repetition of outputs of the second function which means that the function is one-to-one.

1.8.3. EXAMPLE.

Let us compare the functions $y = x^2$ and $y = 3x + 1$. Since the first function repeats the output $y = 4$ for the inputs $x = 2$ and $x = -2$ ($4 = 2^2$ and $4 = (-2)^2$), the function is not one-to-one. There is no repetition of outputs of the second function which means that the function is one-to-one.

The horizontal line test

A *horizontal line* consists of all points which have y -coordinates equal to the same number. The figure below shows the horizontal line consisting of all points having y -coordinate equal to 2. This line could be described by the equation $y = 2$ (in words the equations says that every value of x is related to 2.)



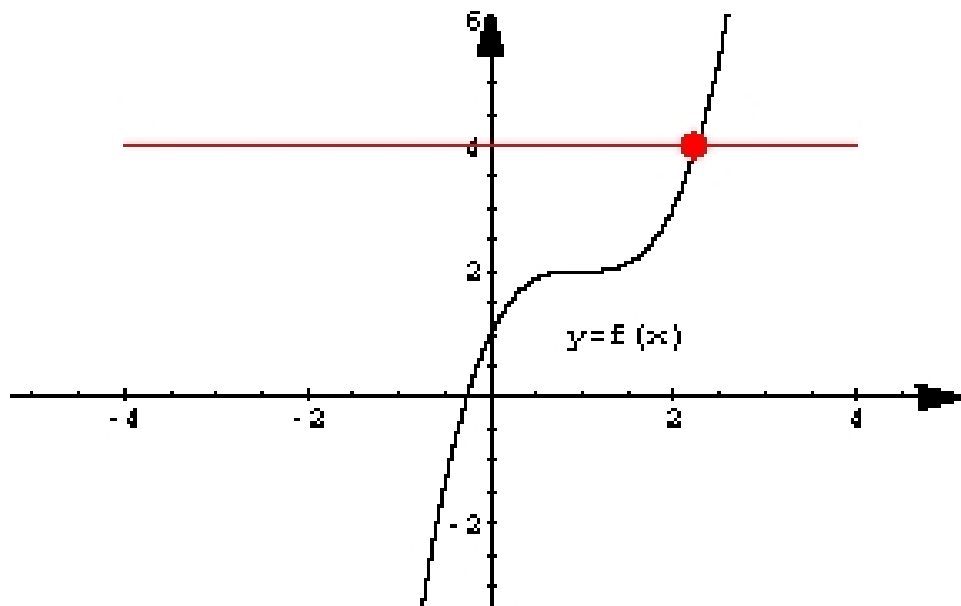
By the definition of an one-to-one function, at most one value of x is related to the given value of y . It follows, that a horizontal line can intersect the graph of a function at most once.

1.8.4. HORIZONTAL LINE TEST.

The graph of a function in a coordinate plane is the graph of an one-to-one function if and only if no horizontal line intersects the graph at more than one point.

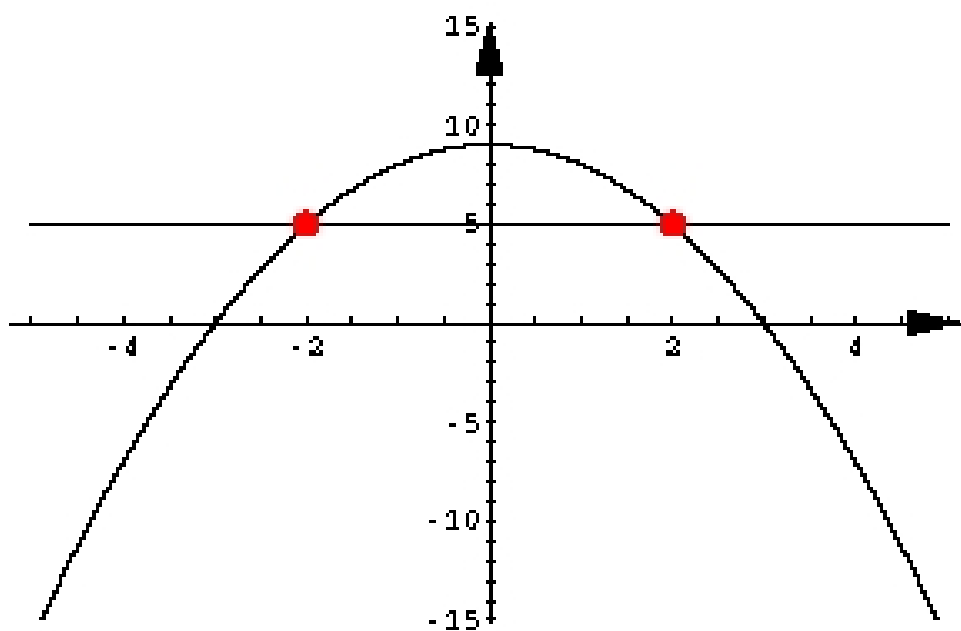
1.8.5. EXAMPLE.

The graph below represents a one-to-one function.



1.8.6. EXAMPLE.

The figure below shows the graph of the function $y = 9 - x^2$. We can see that the graph does not represent a one-to-one function because it has two intersections with the horizontal line $y = 9$.



Identifying one-to-one functions algebraically

1.8.7. EXAMPLE.

We have checked that the function $f(x) = 9 - x^2$ in the Example 1.8.6 is not one-to-one by applying the Horizontal Line Test. Now, we will show how to do it algebraically. First, we need to change the functional notation into an equation in x and y . The substitution $y = f(x)$ give us $y = 9 - x^2$. Now, we solve the equation for the variable x .

$$y = 9 - x^2$$

$$0 = 9 - x^2 - y$$

$$x^2 = 9 - y$$

If $y < 9$ then $9 - y$ is positive and the equation above has two solution for x , $x = \sqrt{9 - y}$ or $x = -\sqrt{9 - y}$. For instance, for $y = 5$ we obtain $x = 2$ or $x = -2$. The inputs 2 and -2 give the same output 5. It means that the function is not one-to-one.

1.8.8. EXERCISES.

1. Exercise. Which of the following functions is one-to-one?

A. $y = 3$

B. $\{(2, 3), (1, 2), (5, 2), (3, 17)\}$

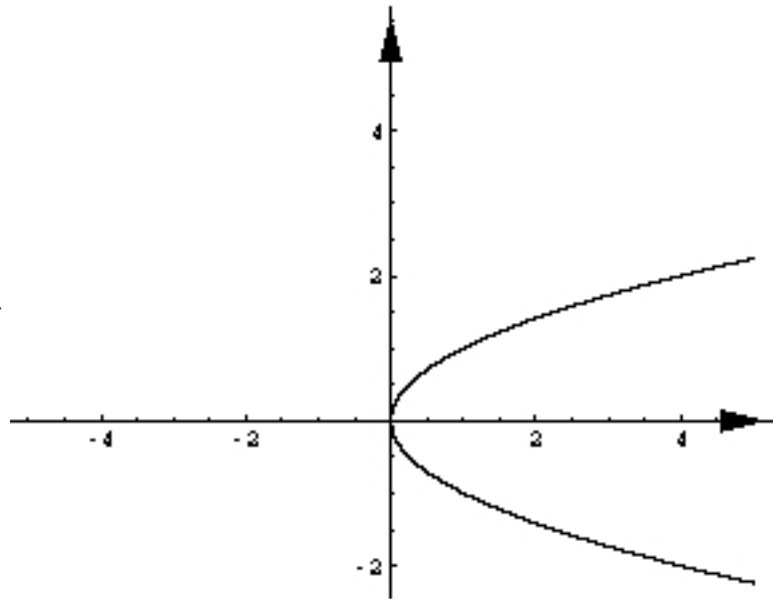
C. $y = |x|$

D. $\{(2, 3), (1, 2), (5, 1), (3, 17)\}$

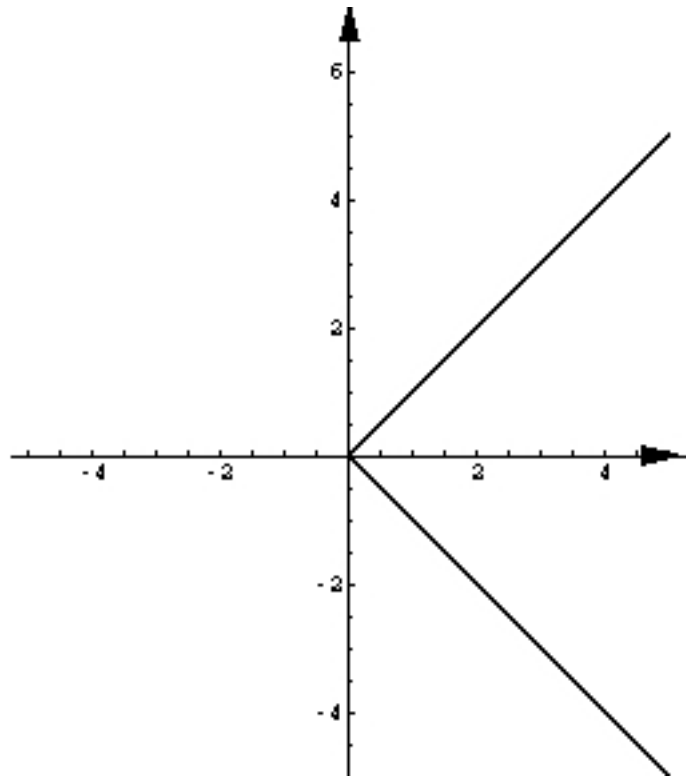
Go to answer 1

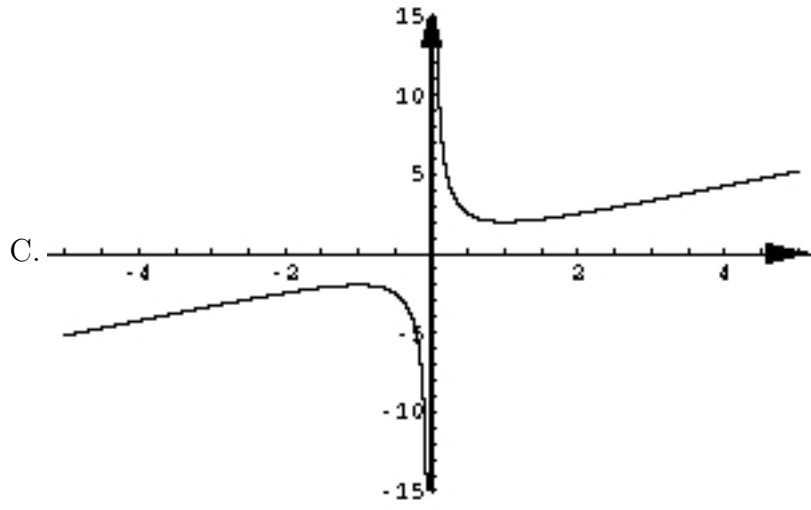
2. Exercise. Which of the following graphs represents a one-to-one function?

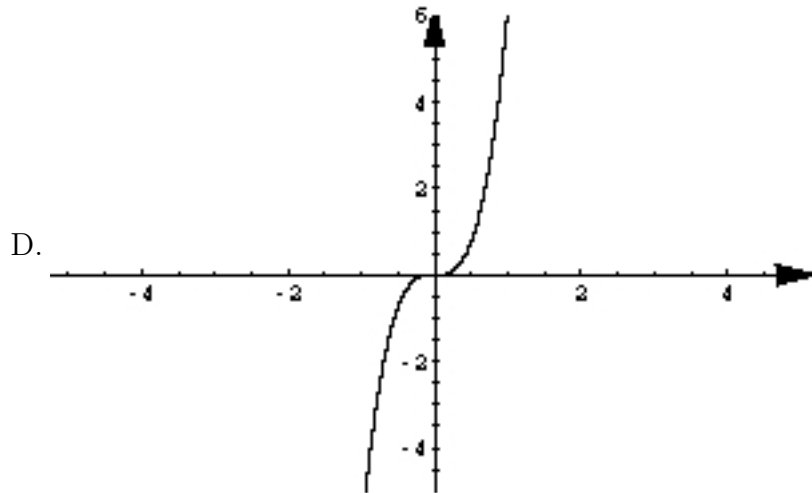
A.



B.







Go to answer 2

3. Exercise. The function given by the equation $y = x^2 - 2x + 1$ is not a one-to-one function because
- A. two is related to one so not one-to-one
 - B. there is one value of x related to two values of y
 - C. if $y = 4$ then the equation $y = x^2 - 2x + 1$ has two solutions $x = -1$ and $x = 3$
 - D. one input gives two different outputs

Go to answer 3

1.8.9. ANSWERS.

1. Answer to Exercise 1 is "D".

Go back 1

2. Answer to Exercise 2 is "D".

Go back 2

3. Answer to Exercise 3 is "C".

Go back 3