

1.9. Inverse relations and inverse functions

Inverse relations

I have a sister. I describe the relation with my sister saying "I am a brother of her" which gives the ordered pair $(me, mysister)$. My sister describes our relation saying "I am a sister of him" and it leads to the ordered pair $(mysister, me)$. The first describes our relation from my point of view and the other describes the same relation from her point of view. The relations like the one between myself and sister and the one between my sister and myself are considered to be inverse and we obtain one from the other by the interchanging the elements in the ordered pairs. Answering the question in the following quiz should help in understanding this concept better.

1.9.1. INVERSE RELATION QUIZ.

Julio, Harry, Mark, Kathy and Adrienne are related in this way that Julio hates Kathy, and Mark and Adrienne hate Harry. How would you describe the inverse relation between them?

- A. Everyone likes everyone.
- B. Julio does not hate Kathy, and Mark and Adrienne do not hate Harry.
- C. Julio likes Kathy, and Mark and Adrienne like Harry.
- D. Kathy is hated by Julio, and Harry is hated by Mark and Adrienne.

Answer: D

Let R be a relation with the domain X and the range Y . So R is a set of pairs (x, y) where x is an element of X and y is an element of Y . The inverse relation R^{-1} of R is the set pairs (y, x) which are obtained from the pairs (x, y) in R by interchanging x and y .

$$(x, y) \Rightarrow (y, x)$$

1.9.2. DEFINITION.

Let R be a relation which is a subset of the set of all pairs

$$\{(x, y) | x \text{ belongs to } X \text{ and } y \text{ belongs to } Y\}$$

The inverse relation R^{-1} of R is the set

$\{(y, x) | (x, y) \text{ belongs to } R\}$

1.9.3. EXAMPLE

Let us consider the relation $R = \{(2, 3), (4, 5), (1, 5), (3, 4)\}$. The inverse relation is the set $R^{-1} = \{(3, 2), (5, 4), (5, 1), (4, 3)\}$. Let us also notice that the domain of $R = \{2, 4, 1, 3\}$ and the range of $R = \{3, 5, 4\}$ but the domain of $R^{-1} = \{3, 5, 4\}$ and the range of $R^{-1} = \{2, 4, 1, 3\}$.

What we have noticed in the Example 1.7.3 is true in general.

1.9.4. INTERCHANGING DOMAIN AND RANGE RULE.

DOMAIN of $R \Rightarrow$ RANGE of R^{-1} RANGE of $R \Rightarrow$ DOMAIN of R^{-1}

Finding inverse relations graphically

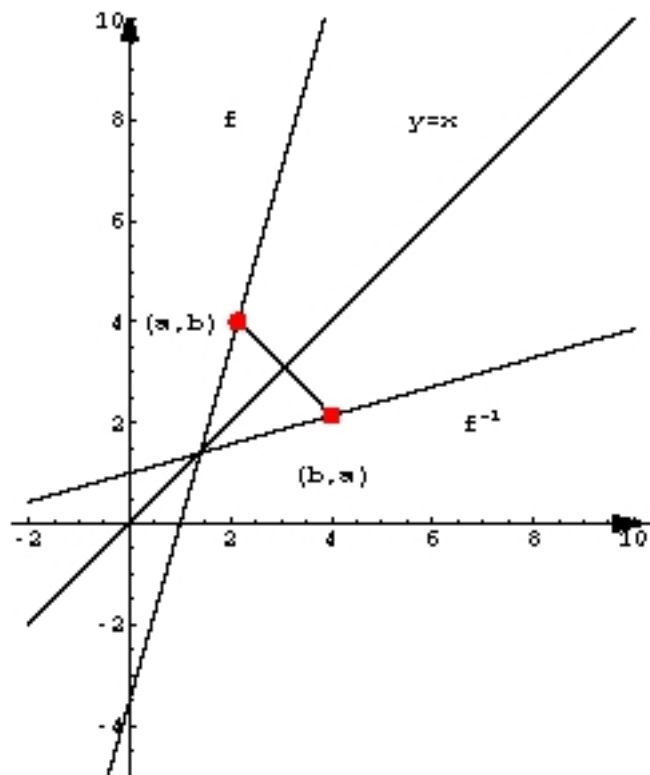
Geometrically, interchanging x and y in ordered pairs (x, y)

$$(x, y) \Rightarrow (y, x)$$

is the reflection in the diagonal line $y = x$. Thus the graph of the inverse relation R^{-1} is the mirror image of the graph of the relation R in the line $y = x$.

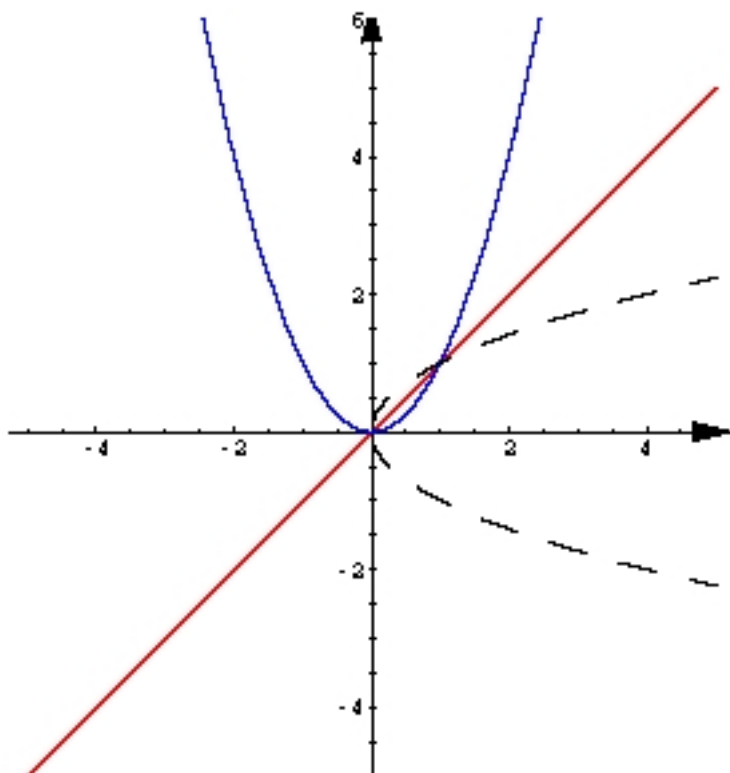
1.9.5. EXAMPLE.

The figure below shows R^{-1} and R where R is a straight line.



1.9.6. EXAMPLE.

The figure below shows R^{-1} and R where R is a parabola.



Finding inverse relations algebraically

Now, let a relation R be given by an equation or an inequality. In order to obtain an equation/inequality describing the inverse relations R^{-1} we replace x by y and y by x everywhere in the original equation/inequality of R .

$$y \Rightarrow x$$

and

$$x \Rightarrow y$$

1.9.7. EXAMPLE.

If R is given by the equation $y = 9 - x^2$ then an equation describing the inverse relation R^{-1} is $x = 9 - y^2$.

1.9.8. EXAMPLE.

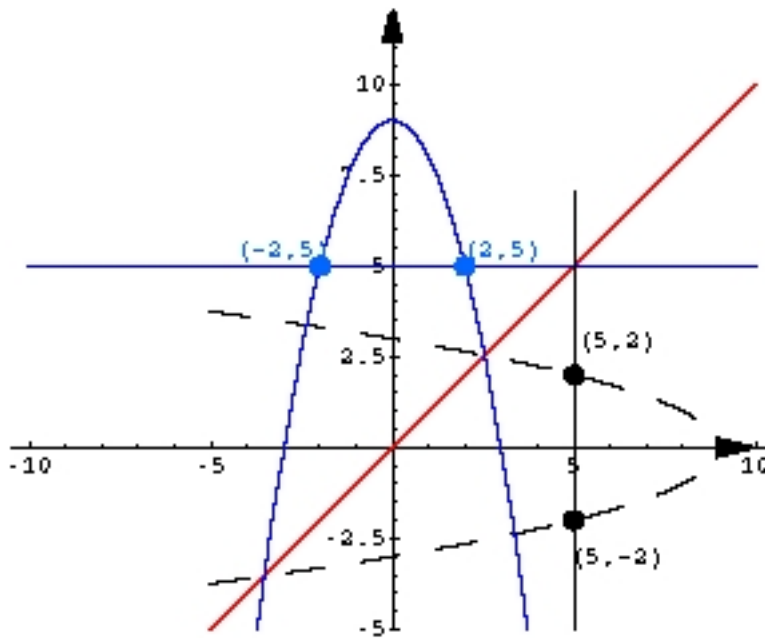
If R is given by the inequality $y < 5x$ then an inequality describing the inverse relation R^{-1} is $x < 5y$.

Inverse functions

Because each function is a relation the inverse relation to a function is well defined. Unfortunately, the inverse relation to a function may be not a function. The next example illustrates it.

1.9.9. EXAMPLE.

Let $f(x) = 9 - x^2$. Then f is a function. By the substitution $y = f(x)$ we obtain the equation $y = 9 - x^2$ which describes a relation between x and y . Now, we replace x by y and y by x everywhere in the equation. It provides the equation $x = 9 - y^2$ which describes the inverse relation. The figure below shows the function f and its inverse relation f^{-1} .



As you can see in the figure, the equation $x = 9 - y^2$ relates $x = 5$ to both $y = 2$ and $y = -2$ so it does not describe a function. Thus the relation f^{-1} is not a function. Let us try to understand it better. In the Example 1.6.5 we checked that the function $f(x) = 9 - x^2$ is not one-to-one by applying the Horizontal Line Test. The horizontal line $y = 5$ shown in the figure intersects the graph of the function f at two points $(2, 5)$ and $(-2, 5)$. The

mirror image of the horizontal line $y = 5$ in the diagonal line $y = x$ is the vertical line $x = 5$. The mirror images of the points $(2, 5)$ and $(-2, 5)$ are the points $(5, 2)$ and $(5, -2)$. So the vertical line $x = 5$ intersects the graph of f^{-1} at two points $(5, 2)$ and $(5, -2)$. Thus for a function which does not satisfy the Horizontal Line Test the inverse does not satisfy the Vertical Line Test and the inverse is not a function. We can conclude that in order for the inverse f^{-1} to be a function the original function f has to be one-to-one.

1.9.10. EXAMPLE.

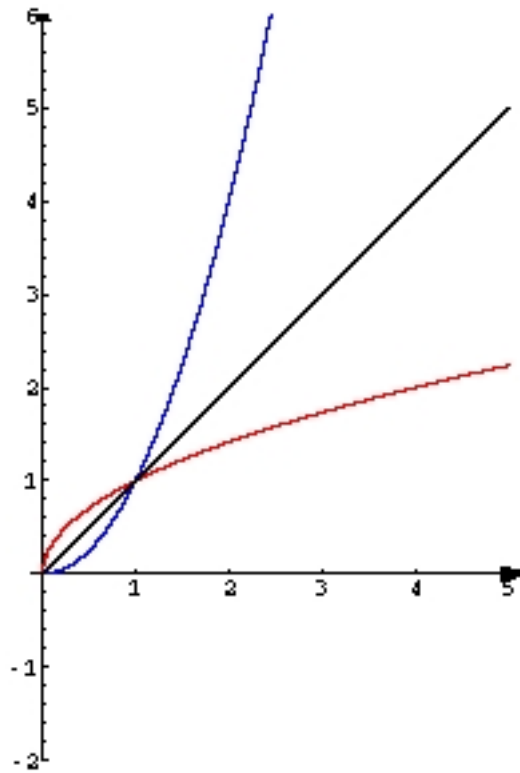
The function $y = x^2$ does not satisfy the Horizontal Line Test over the domain consisting of all real numbers but the function $y = x^2$ satisfies the Horizontal Line Test over the domain $[0, x)$ so the inverse function exists is for $x \geq 0$ and $y \geq 0$. The inverse function satisfies the equation $x = y^2$ for $x \geq 0$ and $y \geq 0$. When we solve the last equation for y it gives us

$$\sqrt{x} = y$$

and

$$\sqrt{x} = y$$

and the graph below illustrates the square root function



1.9.11. STEP BY STEP PROCEDURE FOR FINDING INVERSE FUNCTIONS.

Step 1. In the equation for $f(x)$, replace $f(x)$ by y .

Step 2. Replace x by y and y by x everywhere in equation.

Step 3. Solve the new equation for y .

Step 4. Replace y by $f^{-1}(x)$.