2.2. Zeros of polynomial functions.

In an earlier course, you learned to write polynomials as a product. This process is called *factoring*. The polynomials used in the earlier course were those factorable as polynomials with integer coefficients or they were relatively prime (irreducible over the integers). In this lesson, you will learn to factor polynomials over the real and *complex numbers* but, first let's use the factoring techniques you already learned to solve equations and familiarize ourselves with the terminology.

2.2.1. SPECIAL TOOLS FOR FACTORING:

- *Greatest Common Factor (Monomial factor)*
- *Factor by grouping*
- *Difference of two squares*
- *Sum or Difference of cubes*
- *Factoring trinomials*

2.2.2. EXAMPLE. Given the example, review factoring techniques.

Example:	$P(x) = x^3 - x^2 - 9x + 9$	Polynomial has four terms
	$P(x) = (x^3 - x^2) + (-9x + 9)$	Factor by grouping technique
	$P(x) = x^2(x - 1) - 9(x - 1)$	Greatest Common Factor
	$P(x) = (x^2 - 9)(x - 1)$	Greatest Common Factor
	$P(x) = (x - 3)(x + 3)(x - 1)$	Difference of two squares

Factors are $(x - 3)$, $(x + 3)$, and $(x - 1)$.

Now, to find *roots*, *zeros*, or *solutions*, the polynomial function in factored form must be written as an equation of the form: $P(x) = 0$.

$$P(x) = (x - 3)(x + 3)(x - 1)$$
$$(x - 3)(x + 3)(x - 1) = 0$$

Then apply the *Zero-Factor Property* to write each factor as a linear equation.

Solve each linear equation.

$x - 3 = 0$; $x + 3 = 0$; or $x - 1 = 0$
$x = 3$; $x = -3$ $x = 1$

For the equation $P(x) = 0$:

Its roots are : x = 3; x = -3; x = 1.

Its zeros are : x = 3; x = -3; x = 1.

Its solutions are: x = 3; x = -3; x = 1.

and its *solution set* is: {-3, 1, 3}.

Find out what happens when you evaluate the polynomial function for each value of x that was a solution of the equation $P(x) = 0$.

Let $x = 1$; find $P(1)$	Let $x = 3$; find $P(3)$	Let $x = -3$; find $P(-3)$
$P(1) = 1^3 - 1^2 - 9(1) + 9$	$P(3) = 3^3 - 3^2 - 9(3) + 9$	$P(-3) = (-3)^3 - (-3)^2 - 9(-3) + 9$
$P(1) = 1 - 1 - 9 + 9$	$P(3) = 27 - 9 - 27 + 9$	$P(-3) = -27 - 9 + 27 + 9$
$P(1) = 0$	$P(3) = 0$	$P(-3) = 0$

Your graphing calculator can be most helpful in evaluating polynomials at some given value of x.

Using a TI-82 graphing calculator, follow these steps:

1. Enter the polynomial function in the Y= menu. Use the *X,T,~~Θ~~* to enter the variable x and use the carat (^) to enter the exponent.

EXAMPLE TO ENTER: $P(x) = 3x^2 + x - 10$
KEY IN THE FOLLOWING: 3 *X,T,~~Θ~~* ^ 2 + *X,T,~~Θ~~* - 10

2. Use the CALC feature (2nd TRACE) and select the first menu option, 1: value.

3. The calculator will display EVAL *X* =, prompting you for a number. Enter the x value.

4. The calculator will display the given x value and its corresponding y value.

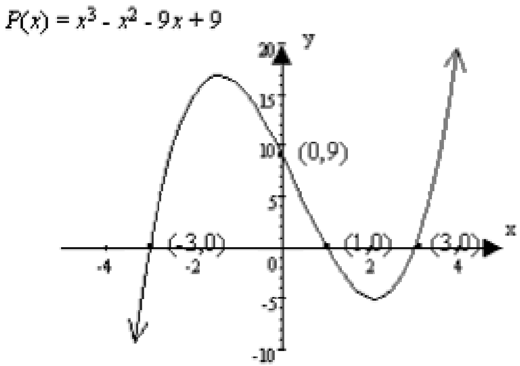
Each of the values is equal to zero. Thus, we have also found the *x-intercepts* of the graph of the polynomial function.

Its x-intercepts are: (1, 0), (3, 0), and (-3, 0).

Evaluate the polynomial $P(x) = x^3 - x^2 - 9x + 9$ for $x = 0$. You will learn that $P(0) = 9$.

The point (0, 9) is the *y-intercept* of the graph of the function.

These *x*- and *y*- intercepts can be plotted to graph the polynomial function. You can draw a smooth curve to pass through these points.



Try to identify a relationship between the degree of the polynomial which is 3, the sign of its leading coefficient, and

- the left and right behavior of the graph and
- the number of times the graph changes direction (turning points).

2.2.3. GUIDED PRACTICE:

Find the factors of the polynomial function $P(x) = x^4 - 18x^2 + 81$.

 Which of the following factoring techniques would you apply?

Factoring trinomials is correct. Of course, you could also use the perfect square trinomial which is a special case of factoring trinomials.

We shall factor the trinomial by setting up two binomial factors.

$$P(x) = (\qquad)\ (\qquad)$$

 What terms will you use in each factor?

How did you do?

$$P(x) = (x^2 - 9)\ (x^2 - 9)$$
or
$$P(x) = (x^2 - 9)^2$$

This factoring is not complete. Notice that the factors are quadratic (2nd degree).
So, we must continue to factor further.

 What other factoring technique could we apply now?

Did you say, "Difference of two squares?"

 That is correct!

Now, set up two more binomial factors for each squared factor.

$$P(x) = (x^2 - 9)\ (x^2 - 9)$$
$$P(x) = (\qquad)\ (\qquad)\ (\qquad)\ (\qquad)$$

The factors of each quadratic factor are *conjugates*. It means that the terms of the two binomials are the same except the signs of one of he terms are additive inverses.

$(x – 3)$ and $(x + 3)$ are conjugates.

Thus, the polynomial in a completely factored form would be

$P(x) = (x - 3)(x + 3)(x - 3)(x + 3)$
or
$P(x) = (x - 3)^2\ (x + 3)^2$

 What are the **factors** of the polynomial function

$P(x) = x^4 – 18x^2 + 81$?

FACTORS: $(x - 3)^2$ and $(x + 3)^2$

Because the same factor(s) occur more than once, the polynomial is said to have *multiplicity*. In this example, each factor has multiplicity of two.

 What are the **roots, zeros,** or **solutions** of the polynomial equation P(x) = 0?

 For quick review:

1) Write the polynomial function in factored form as an equation of the form:

$$P(x) = 0.$$
$$P(x) = (x - 3)(x + 3)(x - 3)(x + 3)$$
$$(x - 3)(x + 3)(x - 3)(x + 3) = 0$$

2) Apply the Zero-Factor Property and write each factor as a linear equation.

$$(x - 3) = 0; \quad (x + 3) = 0; \quad (x - 3) = 0; \quad (x + 3) = 0$$

3) Solve each linear equation.

$$X = 3; \quad x = -3; \quad x = 3; \; x = -3$$

Its roots are: $x = -3$ and $x = 3$

Its zeros are: $x = -3$ and $x = 3$ each having multiplicity of two.

Its solutions are: x = -3 and x = 3

Its solution set is: {-3, 3}

 What are the x- and y- intercepts of the graph of the polynomial function?

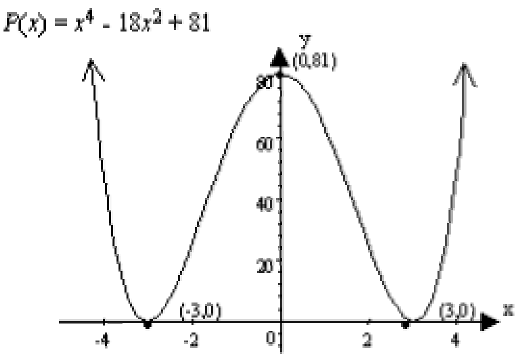
Did you get:
x-intercepts: (-3, 0) and (3, 0) and y-intercept: (0, 81)?

Plot these points and graph the function.

Have you determined any relationship on the degree of the polynomial and the left and right sides of the graph?

What about the turning points? How many turning points are there on this graph?

Does the leading coefficient have any effect on the graph?



2.2.4. (x - k) AS A FACTOR.

If (x - k) is a factor of a polynomial, then

- $x - k = 0$ when the Zero-Factor Property is applied and
- $x = k$ when the linear equation is solved for x.

So, given a factor in the form (x - k), we can say that

- k is a **root** or **zero** of the function $P(x)$ and

k is a **solution** of the equation $P(x) = 0$.

Very important is to be able to express a given root, zero, or solution as a factor.

Given the zero, root, or solution to be $x = 4$,
then, $k = 4$

and the factor in the form of $(x - k)$
would be $(x - 4)$.

Again, given that the zero, root, or solution is $x = -4$
then, $k = -4$

and the factor in the form of $(x - k)$
would be $(x - (-4))$ or $(x + 4)$ when simplified.

Various approaches can be used to determine if $(x - k)$ is a factor of a polynomial.

- Factoring
- Long Division
- Synthetic Division
- Factor Theorem

Factoring:

Trying to factor polynomials of higher degrees may be somewhat difficult. If YOU can easily factor the polynomial, then factor it completely and compare your results with the given factor.

Example:
Is $(x - 5)$ a factor of $x^2 - x - 15$?

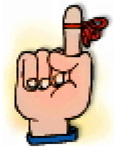
Factoring this trinomial gives us $(x - 5)(x + 3)$as factors.

Thus, $(x - 5)$ is a factor of $x^2 - x - 15$.

Is $(x - 5)$ a factor of $2x^3 - 5x^2 - 28x + 15$?

Factoring this polynomial expression may not be as easy. So, we must use other techniques.

Long Division:

 Remember to write the polynomial in **standard form** with exponents in **descending order.**

Example:
Let's determine if (x + 2) is a factor of $6x^3 + 13x^2 - 10x - 24$.

In the next page you will see the process of long division applied to this problem.

$$\frac{6x^3+13x^2-10x-24}{x+2} = x+2 \overline{\big) 6x^3+13x^2-10x-24}$$

1st term of partial quotient:

Divide 1st term of dividend by 1st term of divisor

$$\frac{6x^3}{x} = 6x^2$$

Multiply partial quotient $6x^2$ by divisor :

$6x^2(x+2) = 6x^3 + 12x^2$

Subtract the two polynomials and bring down next term

$$\begin{array}{r} 6x^2 \qquad\qquad\qquad\quad \\ x+2 \overline{\big) 6x^3+13x^2-10x-24} \\ \underline{\mp 6x^3 \mp 12x^2} \qquad\qquad\quad \\ x^2-10x \qquad\quad \end{array}$$

2nd term of partial quotient

Again, divide 1st term of dividend by 1st term of divisor

$$\frac{x^2}{x} = x$$

Multiply partial quotient x by divisor :

$x(x+2) = x^2 + 2x$

Subtract the two polynomials and bring down the next term

$$\begin{array}{r} 6x^2 + x \qquad\qquad \\ x+2 \overline{\big) 6x^3+13x^2-10x-24} \\ \underline{\mp 6x^3 \mp 12x^2} \qquad\qquad\quad \\ x^2-10x \qquad\quad \\ \underline{\mp x^2 \mp 2x} \qquad\quad \\ -12x-24 \end{array}$$

3rd term of partial quotient

Again, divide 1st term of dividend by 1st term of divisor

$$\frac{-12x}{x} = -12$$

Multiply partial quotient - 12 by divisor :

$-12(x+2) = -12x - 24$

Subtract the two polynomials

$$\begin{array}{r} 6x^2 + x - 12 \qquad \\ x+2 \overline{\big) 6x^3+13x^2-10x-24} \\ \underline{\mp 6x^3 \mp 12x^2} \qquad\qquad\quad \\ x^2-10x \qquad\quad \\ \underline{\mp x^2 \mp 2x} \qquad\quad \\ -12x-24 \\ \underline{\pm 12x \pm 24} \\ 0 \text{ remainder} \end{array}$$

Because the remainder of the division process is zero, then $(x + 2)$ is a factor of the polynomial $6x^3 + 13x^2 - 10x - 24$. The product of the divisor and quotient polynomials represent the polynomial in factored form.

$$P(x) = (x + 2)(6x^2 + x - 12).$$

NOTICE:

- (x + 2) written in the form (x - k) would be (x - (- 2)) in which the value of k would be (-2).
- ($6x^2 + x - 12$) is not a linear factor. It is a quadratic factor and can be factored further using our factoring skills.
- Thus, $6x^2 + x - 12 = (2x + 3)(3x - 4)$

The completely factored polynomial in factored form would be $P(x) = (x + 2)(2x + 3)(3x - 4)$.

The **ROOTS** of the function are $x = -2$; $x = -3/2$; $x = 4/3$.

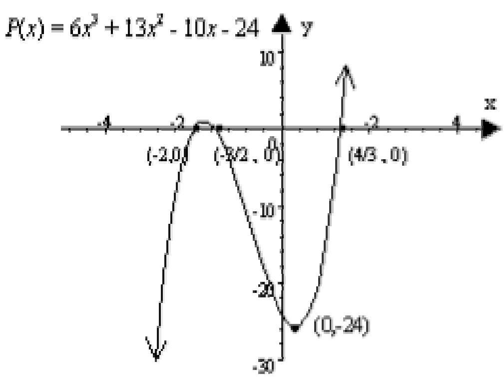
The **ZEROS** of the function are $x = -2$; $x = -3/2$; $x = 4/3$.

The **SOLUTIONS** of the equation $P(x) = 0$ are $x = -2$; $x = -3/2$; $x = 4/3$.

The **x-intercepts** of the graph of the function are (- 2, 0); (-3/2, 0); (4/3, 0).

The **y-intercept** of the graph of the function is (0, -24).

The **GRAPH** of the function is:



$$\frac{6x^3+13x^2-10x-24}{x+2} = x+2\overline{)6x^3+13x^2-10x-24}$$

1st term of partial quotient:

Divide 1st term of dividend by 1st term of divisor

$$\frac{6x^3}{x} = 6x^2$$

$$\begin{array}{r} 6x^2 \\ x+2\overline{)6x^3+13x^2-10x-24} \\ \underline{\mp 6x^3 \mp 12x^2} \\ x^2 - 10x \end{array}$$

Multiply partial quotient $6x^2$ by divisor :

$6x^2(x+2) = 6x^3 + 12x^2$

Subtract the two polynomials and bring down next term

2nd term of partial quotient

Again, divide 1st term of dividend by 1st term of divisor

$$\frac{x^2}{x} = x$$

Multiply partial quotient x by divisor :

$x(x+2) = x^2 + 2x$

Subtract the two polynomials and bring down the next term

$$\begin{array}{r} 6x^2 + x \\ x+2\overline{)6x^3+13x^2-10x-24} \\ \underline{\mp 6x^3 \mp 12x^2} \\ x^2 - 10x \\ \underline{\mp x^2 \mp 2x} \\ -12x-24 \end{array}$$

3rd term of partial quotient

Again, divide 1st term of dividend by 1st term of divisor

$$\frac{-12x}{x} = -12$$

Multiply partial quotient - 12 by divisor :

$-12(x+2) = -12x - 24$

Subtract the two polynomials

$$\begin{array}{r} 6x^2 + x - 12 \\ x+2\overline{)6x^3+13x^2-10x-24} \\ \underline{\mp 6x^3 \mp 12x^2} \\ x^2 - 10x \\ \underline{\mp x^2 \mp 2x} \\ -12x - 24 \\ \underline{\pm 12x \pm 24} \\ 0 \text{ remainder} \end{array}$$

Because the remainder of the division process is zero, then $(x + 2)$ is a factor of the polynomial $6x^3 + 13x^2 - 10x - 24$.

The product of the divisor and quotient polynomials represent the polynomial in factored form.

$$P(x) = (x + 2)(6x^2 + x - 12).$$

NOTICE:

- (x + 2) written in the form (x - k) would be (x - (- 2)) in which the value of k would be (-2).
- ($6x^2$ + x - 12) is not a linear factor. It is a quadratic factor and can be factored further.
- Thus, $6x^2 + x - 12 = (2x + 3)(3x - 4)$

The polynomial in a completely factored form would be

$$P(x) = (x + 2)(2x + 3)(3x - 4).$$

The **ROOTS** of the function are $x = -2$; $x = -3/2$; $x = 4/3$.

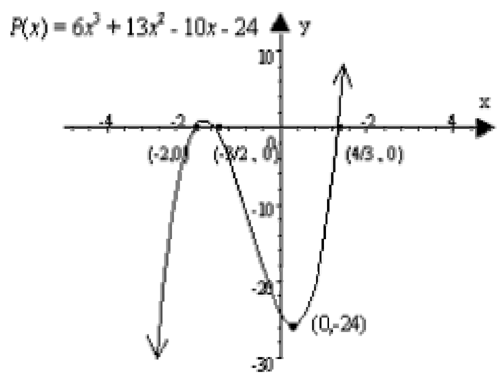
The **ZEROS** of the function are $x = -2$; $x = -3/2$; $x = 4/3$.

The **SOLUTIONS** of the equation $P(x) = 0$ are $x = -2$; $x = -3/2$; $x = 4/3$.

The ***x*-intercepts** of the graph of the function are:
(- 2, 0); (-3/2, 0); (4/3, 0).

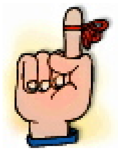
The ***y*-intercept** of the graph of the function is (0, -24).

The **GRAPH** of the function is:



2.2.5. SYNTHETIC DIVISION:

Synthetic division is a shortened form of long division. This process involves only the coefficients of the *dividend* and the zero of the *divisor*. The divisor should be a linear expression in the form of (x - k).

 Remember to write the polynomial in **standard form** with exponents in **descending order**.

Example:

Let's determine if (x - 5) is a factor of $2x^3 - 11x^2 + 2x + 15$.

$$\frac{2x^3 - 11x^2 + 2x + 15}{x - 5} = x - 5\overline{)2x^3 - 11x^2 + 2x + 15}$$

Coefficients of dividend : $\;\; \begin{array}{|cccc}2 & -11 & 2 & 15 \\ \hline \end{array}$

Zero of the divisor : $x - 5 = 0$, then $x = 5$.

Thus, 5 is the zero of the divisor.

$$\begin{array}{lr|rrrr} & 5 & 2 & -11 & 2 & 15 \\ \hline \text{Product row.} & & & 10 & -5 & -15 \\ \hline \text{Bottom row.} & & 2 & -1 & -3 & 0 \end{array}$$

1. Drop first coefficient to the bottom row.
2. Multiply the divisor times each number in the bottom row. Place each product in the . product row
3. Add each column and place sum in the bottom row.

REMEMBER, Synthetic Division is a process involving only coefficients.

Therefore, 2; -1; -3 and 0 found on the bottom row of the synthetic division represent the coefficients of the *quotient* where 0 (the last number) represents the remainder.

The quotient would be expressed as $\mathbf{2}x^2$ - $\mathbf{1}x$ - $\mathbf{3}$ with a zero remainder or

$$\frac{2x^3 - 11x^2 + 2x + 15}{x - 5} = 2x^2 - x - 3 + \frac{0}{x - 5}$$

Notice that the degree of the quotient is <u>one less than the degree of the dividend</u>.

The quotient is sometimes referred to as the depressed polynomial.

The remainder of this division process is zero.

What does it mean when the remainder is zero?

WHEN A REMAINDER IS ZERO, WE KNOW MANY THINGS..

1) The divisor **is a factor** of the dividend.

From our example we know that $(x - 5)$ is a factor of
$2x^3 - 11x^2 + 2x + 15$.

2) From the factor we know that:

$x = 5$ is a solution of the equation $P(x) = 0$.
$x = 5$ is a root of the polynomial function.
$x = 5$ is a zero of the polynomial function.

3) The polynomial can be expressed in factored form as the product of divisor and quotient.

$P(x) = (x - 5)(2x^2 - x - 3)$

Because one of the factors is quadratic, one should attempt to factor further over the integers. This quadratic is factorable over the integers. $(2x^2 - x - 3) = (2x - 3)(x + 1)$

Thus, the polynomial function when factored completely is:

$P(x) = (x - 5)(2x - 3)(x + 1)$.

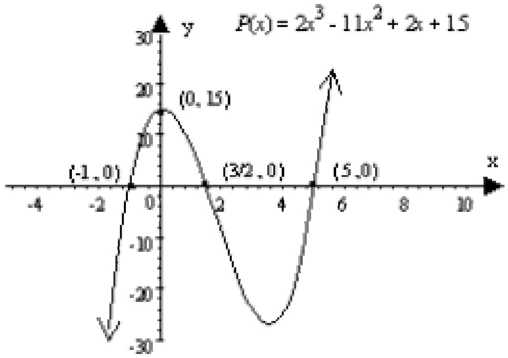
So what do we know about **this** polynomial function?

Let's see:

Given: $P(x) = 2x^3 - 11x^2 + 2x + 15$ or in factored form $P(x) = (x - 5)(2x - 3)(x + 1)$

then,

- $x = 5$; $x = 3/2$; and $x = -1$ are all **zeros** of the polynomial,
- $(5, 0)$; $(3/2, 0)$; and $(-1, 0)$ are **x-intercepts** of the graph of the polynomial,
- $(0, 15)$ is the ***y*-intercept** of the graph of the polynomial,
- and the graph of the polynomial function is:



2.2.6. FACTOR THEOREM.

Given a factor in the form of $(x - k)$, one can determine if it is a factor of a polynomial by simply evaluating the polynomial function at $x = k$.

A polynomial $P(x)$ has a factor $(x - k)$ if and only if $P(k) = 0$.

Example:

Let's determine if $(x + 3)$ is a factor of $2x^4 + 9x^3 - 4x^2 - 12x + 21$?

Given $(x + 3)$, then $k = -3$.

Another way to determine the value of k is to solve for x by setting the linear factor equal to zero.

Let $x + 3 = 0$ then,
$x = -3$ and since $x = k$,
$k = -3$.

Now, evaluate the polynomial $P(x) = 2x^4 + 9x^3 - 4x^2 - 12x + 21$ at $x = k$.

MANUALLY:

$P(k) = 2k^4 + 9k^3 - 4k^2 - 12k + 21$ and $k = -3$

$P(-3) = 2(-3)^4 + 9(-3)^3 - 4(-3)^2 - 12(-3) + 21$

$P(-3) = 2(81) + 9(-27) - 4(9) - 12(-3) + 21$

$P(-3) = 162 - 243 - 36 + 36 + 21$

$P(-3) = -60$

USING A TI-82 GRAPHING CALCULATOR:

1. Enter the function in Y=
2. Use CALC feature
3. Select menu-option 1: value
4. Enter -3

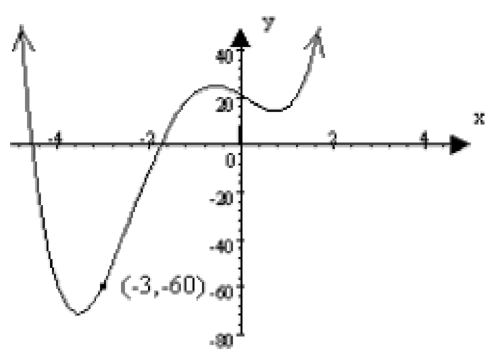
Solution is displayed: $x = -3$ $y = -60$

$P(k)$ does not equal 0!

What does that mean?

It means that $(x + 3)$ is **not** a factor of $2x^4 + 9x^3 - 4x^2 - 12x + 21$.

But we do know that the point (-3, -60) is a point on the graph of the polynomial function.



It is interesting to note that the remainder of synthetic division is always equal to the result when the polynomial function is evaluated at $x = k$. The remainder is always equal to the result of $P(k)$.

2.2.6. REMAINDER THEOREM. If the polynomial $P(x)$ is divided by $x - k$, then the remainder is equal to $P(k)$. Thus, we can use synthetic division to evaluate a polynomial function at $x = k$.

2.2.7. EXAMPLE.

Divide $(-2x^5 + x^3 - 6x^2 - 2x + 13) \div (x + 2)$:

Use synthetic division. (Remember to include a zero for the coefficient of x^4 since it does not appear in the polynomial .)

$k = -2$

$$\begin{array}{r|rrrrrr} -2 & -2 & 0 & 1 & -6 & -2 & 13 \\ \hline & & 4 & -8 & 14 & -16 & 36 \\ \hline & -2 & 4 & -7 & 8 & -18 & 49 \end{array}$$

Thus, $\dfrac{-2x^5 + x^3 - 6x^2 - 2x + 13}{x + 2} = -2x^4 + 4x^3 - 7x^2 + 8x - 18 + \dfrac{49}{x + 2}$

Now evaluate $P(-2)$:

$$P(x) = -2x^5 + x^3 - 6x^2 - 2x + 13$$
$$P(k) = -2k^5 + k^3 - 6k^2 - 2k + 13$$
$$P(-2) = -2(-2)^5 + (-2)^3 - 6(-2)^2 - 2(-2) + 13$$
$$P(-2) = -2(-32) + (-8) - 6(4) - 2(-2) + 13$$
$$P(-2) = 64 - 8 - 24 + 4 + 13$$
$$P(-2) = 49$$

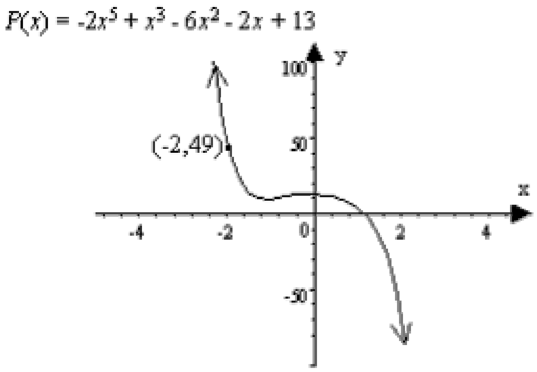
USING A TI-82 GRAPHING CALCULATOR:

1. Enter the function in Y=
2. Use CALC feature
3. Select menu-option 1: value
4. Enter -2

Solution is displayed: x = -2 y = 49


What do you know about this example?

From this example, we know that (x + 2) is <u>not</u> a factor of $-2x^5 + x^3 - 6x^2 - 2x + 13$. We also know that (-2, 49) is a point on the graph of the polynomial.



2.2.7. COMPLEX ZEROS.

Some zeros of a polynomial may be *complex numbers*. To apply synthetic division to zeros that are complex numbers requires that you know how to simplify complex numbers.


Remember that $i = \sqrt{-1}$ and for any positive number *b*, $\sqrt{-b} = i\sqrt{b}$. Also, remember to express your answer in the form *a* + *bi* .

FOR A QUICK REVIEW:

Operation	General Rule	Example
Addition	*a* + *bi* + *c* + *di* = (*a* + *c*) + (*b* + *d*)i	3 + 5*i* + 7 + 4*i* = (3 + 7) + (5 + 4)*i* = 10 + 9*i*
Subtraction	*a* + *bi* - (*c* + *di*) = *a* + *bi* - *c* - *di* = (*a* -*c*) + (*b* - *d*)i	3 + 5*i* - (7 + 4*i*) = 3 + 5*i* - 7 - 4*i* = (3 + 7) + (5 + 4)*i* = 10 + 9*i*
Multiplication **Note:** $i^2 = -1$	(*a* + *bi*)(*c* + *di*) = *ac* + a*di* + *cbi* + bdi^2 = *ac* + (a*d* + *cb*)*i* + *bd*(-1) = (*ac* - *bd*) + (a*d* + *cb*)*i*	(3 + *i*)(4 - 2*i*) = 12 - 6*i* + 4*i* - $2i^2$ = 12 - 2*i* - 2(-1) = 14 - 2*i*

Division Note: Rationalize Denominator	$\frac{a}{b+ci} = \frac{a}{b+ci} \cdot \frac{b-ci}{b-ci}$ $= \frac{ab-aci}{b^2-c^2i^2}$ $= \frac{ab-aci}{b^2-c^2(-1)}$ $= \frac{ab-aci}{b^2+c^2}$	$\frac{2}{3+4i} = \frac{2}{3+4i} \cdot \frac{3-4i}{3-4i}$ $= \frac{2(3)-2(4)i}{3^2-4^2i^2}$ $= \frac{6-8i}{9-16(-1)}$ $= \frac{6-8i}{25}$ $= \frac{6}{25} - \frac{8}{25}i$

Let's determine if $(x + i)$ is a factor of $f(x) = 3x^3 + x^2 + 3x + 1$.

Which process should we use?

Synthetic division?

So, if $(x + i)$ is a factor, then $k = -i$ and

$$\begin{array}{r|rrrr} -i & 3 & 1 & 3 & 1 \\ & & -3i & -3-i & -1 \\ \hline & 3 & 1-3i & -i & 0 \end{array}$$

Since the remainder is zero, we know that $(x + i)$ is a factor of $f(x) = 3x^3 + x^2 + 3x + 1$, and
$x = -i$ is a zero of the function.

Complex zeros occur in pairs known as conjugates.

That is:

if $a + bi$, where $b>0$, is a zero of the polynomial function $P(x)$, then $a - bi$ is also a zero of that function.

So, if $x = -i$ is a zero of $f(x) = 3x^3 + x^2 + 3x + 1$, then $x = i$ is also a zero. Thus, $(x - i)$ is also a factor.

NOTICE: $f(x) = 3x^3 + x^2 + 3x + 1$ and,
$f(x) = (x + i)(3x^2 + (1 - 3i)x - i)$
One factor is linear and the other is quadratic. We can use the quadratic factor to divide by $(x - i)$.
By using synthetic division again, we can determine the third factor.

Let's see. For $(x - i)$ we have $k = i$.

$$\begin{array}{r|rrr} i & 3 & 1-3i & -i \\ & & 3i & i \\ \hline & 3 & 1 & 0 \end{array}$$

So in factored form, $f(x) = 3x^3 + x^2 + 3x + 1$ and,
$f(x) = (x + i)(x - i)(3x + 1)$.

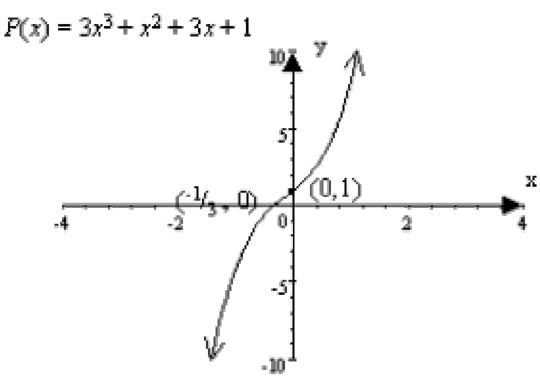
Zeros of the function are $x = -i$, $x = i$, and $x = -{}^1/_3$.

How many x - intercept(s) does this function have?

Guess what!

$(-{}^1/_3, 0)$ is the only x- intercept because it is the only real zero. The other two are imaginary or complex zeros.

Observe what happens when the function is graphed.



To think about??????

Given a cubic polynomial and one zero, the other two zeros can be determined because the quotient or depressed polynomial, after using synthetic division, is quadratic. What must be true about the given zero of a polynomial of fourth (4^{th}) degree in order to find the other three zeros?