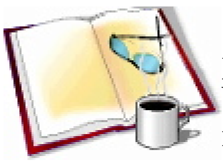
2.3. Finding polynomial functions.

An Introduction:

As is usually the case when learning a new concept in mathematics, the new concept is the reverse of the previous one. Remember how you first learned addition and then you learned subtraction or first you learned multiplication and then you learned division. Each operation being the opposite of the other. Yet another example is one in which you learned the properties of exponents and then you learned about radicals or rational exponents.

Well, finding polynomials is the reverse of finding factors. In the previous lesson, you were given a polynomial and asked to find its factors and zeros. In this lesson, you will be given factors or zeros and asked to find the polynomial of lowest degree with real coefficients.

Example 1: Given the factors $(x - 3)^2(2x + 5)$, find the polynomial of lowest degree with real coefficients.Let's analyze what we already know. $(x - 3)^2$ is a repeated factor, thus the zero 3 has multiplicity of two.



Did you know that the *linear factorization theorem* states that a polynomial of degree *n* has precisely *n* linear factors. And since we have been given three linear factors, the lowest degree of this polynomial must be three.

STEPS	EXAMPLE
Rewrite the factors to show the repeating factor.	$(x - 3)(x - 3)(2x + 5)$
Write factors in polynomial notation. The a_n is the leading coefficient of the polynomial	$P(x) = a_n(x - 3)(x - 3)(2x + 5)$
Multiply $(x - 3)(x - 3)$	$P(x) = a_n(x^2 - 6x + 9)(2x + 5)$
Then multiply $(x^2 - 6x + 9)(2x + 5)$	$P(x) = a_n(2x^3 - 7x^2 + 8x + 45)$
For right now let $a_n = 1$	$P(x) = 2x^3 - 7x^2 + 8x + 45$

Thus, the polynomial of lowest degree with real coefficients, and factors $(x - 3)^2(2x + 5)$ is:

$P(x) = 2x^3 - 7x^2 + 8x + 45$.

So, what did you learn from this example?

The main concept is simply to multiply the factors in order to determine the polynomial of lowest degree with real coefficients.

Example 2:

Find the polynomial of lowest degree with real coefficients given the zeros of -2, $^3/5$, and 4 (multiplicity of 3).


What is the lowest degree of the polynomial?

say 5? 

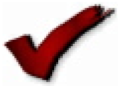
Let's analyze what we already know.
The polynomial is of degree 5

- since $x = -2$ occurs once,
- $x = {}^3/5$ occurs once,
- and $x = 4$ occurs three times.

Remember, a polynomial of degree *n* has at most *n* distinct zeros.
Write the zeros as factors for $x = -2$, $x = {}^3/5$, $x = 4$ (multiplicity of 3).
How are factors written? Look at the steps below for the zero, $x = {}^3/5$.

STEPS	EXAMPLE
1. Take $x = {}^3/5$ multiply both expressions by the denominator 5.	$5(x) = {}^3/5\ (5)$ $5x = 3$
2. Do the additive inverse of 3 to get the expression on the right equal to zero.	$5x - 3 = 3 - 3$ $5x - 3 = 0$
3. Write the left expression in parenthesis as a factor	$(5x - 3)$

Write the other four zeros as factors.

 YOUR ANSWER:
All five factors are $(x + 2)(5x - 3)(x - 4)(x - 4)(x - 4)$.
Write the factors in polynomial notation.

$P(x) = a_n(x + 2)(5x - 3)(x - 4)(x - 4)(x - 4)$

where a_n is the leading coefficient of the polynomial.
Multiply all factors to get the polynomial with real coefficients.

STEPS	EXAMPLE
Multiply two factors at a time. Start by multiplying $(x + 2)(5x - 3)$.	$P(x) = a_n(x + 2)(5x - 3)(x - 4)(x - 4)(x - 4)$ $P(x) = a_n(5x^2 + 7x - 6)(x - 4)(x - 4)(x - 4)$
Multiply the two factors $(5x^2 + 7x - 6)(x - 4)$ or $(x - 4)(5x^2 + 7x - 6)$ since multiplication is *commutative.*	$P(x) = a_n(5x^3 - 13x^2 - 34x + 24)(x - 4)(x - 4)$
Multiply the two factors $(5x^3 - 13x^2 - 34x + 24)(x - 4)$	$P(x) = a_n(5x^4 - 33x^3 + 18x^2 + 160x - 96)(x - 4)$
Multiply the two factors $(5x^4 - 33x^3 + 18x^2 + 160x - 96)(x - 4)$	$P(x) = a_n(5x^5 - 53x^4 + 150x^3 + 88x^2 - 736x + 384)$
Let $a_n = 1$	$P(x) = 5x^5 - 53x^4 + 150x^3 + 88x^2 - 736x + 384$

Thus, the polynomial of lowest degree with real coefficients and zeros of -2, $^3/5$, and 4 (multiplicity of 3) is:

$$P(x) = 5x^5 - 53x^4 + 150x^3 + 88x^2 - 736x + 384.$$

What happens when the zeros are complex numbers?

Example 3:

Given the zeros of $^{-2}/3$ and $5i$, find the polynomial of lowest degree with real coefficients.

 What is the lowest degree of the polynomial?

Did you say 3? 

Since the polynomial will have real coefficients, the complex number $-5i$ is also a zero.

Write the zeros of $^{-2}/_{3}$, $5i$ and $-5i$ as factors in polynomial notation.

$$P(x) = a_n(3x + 2)(x - 5i)(x + 5i)$$

Multiply these factors.

STEPS	EXAMPLE
Multiply the conjugates first because the product is the difference of two squares.	$P(x) = a_n(3x + 2)(x - 5i)(x + 5i)$ $P(x) = a_n(3x + 2)(x^2 + 25)$
Multiply the two factors $(3x + 2)(x^2 + 25)$	$P(x) = a_n(3x^3 + 2x^2 + 75x + 50)$
Let $a_n = 1$	$P(x) = 3x^3 + 2x^2 + 75x + 50$

Thus, the polynomial of lowest degree with real coefficients and zeros of $^{-2}/_{3}$ and $5i$ is:

$P(x) = 3x^3 + 2x^2 + 75x + 50$.

Were you wondering about the leading coefficient a_n?

The next example will illustrate how to use a_n to find the polynomial of lowest degree with real coefficients and meets certain conditions.

Example 4:

Find a polynomial function of lowest degree with real coefficients that has zeros of -4, 0, and $^{2}/_{5}$ and <u>$P(1) = 10$</u>. The graph of this polynomial will pass through the point (1, 10).

The graph of the polynomial with leading coefficient equal to 1 will be compared to the one that meets the specific conditions of <u>$P(1) = 10$</u>.

Write the polynomial with the given zeros in factored form. Do not forget to use a_n as its leading coefficient.

Does your polynomial look like this?

$P(x) = a_n(x + 4)(x)(5x - 2)$

Here's how to derive the factors.

zero: $x = -4$	zero: $x = 0$	zero: $x = {}^2/5$
$x + 4 = -4 + 4$	$x = 0$	$(5)x = {}^2/5\ (5)$
$x + 4 = 0$	$x = 0$	$5x = 2$
factor: $(x + 4)$	factor: (x)	$5x - 2 = 2 - 2$
		$5x - 2 = 0$
		factor: $(5x - 2)$

Multiply the factors.

What is your answer?

Answer: $P(x) = a_n(5x^3 + 18x^2 - 8x)$

Let the leading coefficient $a_n = 1$.

The polynomial of lowest degree with real coefficients will be $P(x) = 5x^3 + 18x^2 - 8x$.

NOW! to meet the second condition that $P(1) = 10$, one must substitute the known values into either polynomial equation, the one with real coefficients $P(x) = a_n(5x^3 + 18x^2 - 8x)$ or the one in factored form $P(x) = a_n(x + 4)(x)(5x - 2)$ to find the value of a_n.

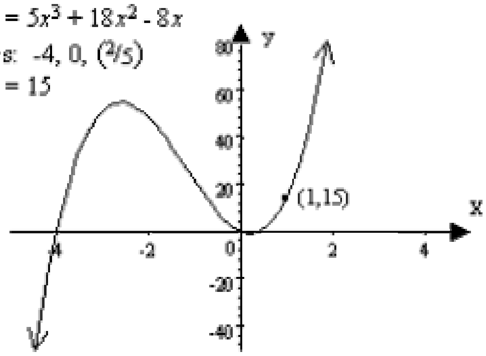
STEPS	EXAMPLE with real coefficients	EXAMPLE in factored form
Substitute $P(x)$ with the value of 10 since $P(1) = 10$.	$P(x) = a_n(5x^3 + 18x^2 - 8x)$ $10 = a_n(5x^3 + 18x^2 - 8x)$	$P(x) = a_n(x + 4)(x)(5x - 2)$ $10 = a_n(x + 4)(x)(5x - 2)$
Substitute all x with the value of 1 since $P(x) = P(1)$	$10 = a_n\big(5(1)^3 + 18(1)^2 - 8(1)\big)$	$10 = a_n(1 + 4)(1)(5(1) - 2)$
Simplify the polynomial expression	$10 = a_n(15)$	$10 = a_n(5)(1)(3)$ $10 = a_n(15)$
Solve for a_n by taking the multiplicative inverse of 15	$^{10}/_{15} = a_n$ or $a_n = {}^{2}/_{3}$	$^{10}/_{15} = a_n$ or $a_n = {}^{2}/_{3}$
Before we let $a_n = 1$. Now, let $a_n = {}^{2}/_{3}$	$P(x) = ({}^{2}/_{3})(5x^3 + 18x^2 - 8x)$	$P(x) = ({}^{2}/_{3})(x + 4)(x)(5x - 2)$ $P(x) = ({}^{2}/_{3})(5x^3 + 18x^2 - 8x)$
Simplify the polynomial	$P(x) = ({}^{10}/_{3})x^3 + 12x^2 - ({}^{16}/_{3})\ x$	$P(x) = ({}^{10}/_{3})x^3 + 12x^2 - ({}^{16}/_{3})\ x$

Thus, the polynomial of lowest degree with real coefficients with zeros of -4, 0, and $^{2}/_{5}$ and $P(1) = 10$ is:

$$P(x) = ({}^{10}/_{3})x^3 + 12x^2 - ({}^{8}/_{3})\ x$$

COMPARE THE GRAPHS OF THE TWO POLYNOMIALS.

$P(x) = 5x^3 + 18x^2 - 8x$
Zeros: -4, 0, ($^2/_5$)
$P(1) = 15$



$P(x) = (^{10}/_3)x^3 + 12x^2 - (^{16}/_3)x$
Zeros -4, 0, $^2/_5$
$P(1) = 10$