

2.5. Graphs of polynomial functions.

In the following lesson you will learn to sketch graphs by understanding what controls their behavior. More precise graphs will be developed in the next two lessons when rational and irrational zeros are introduced. For clarity, a sketch is a rough drawing of the graph. The lesson will assist you in the development of more accurate graphs.

A quiz will be used to review the following prerequisite concepts.

1. Classification of polynomial functions as *linear, quadratic, cubic, or polynomials of higher order.*
2. Linear factorization theorem.
3. Leading coefficient.
4. Evaluating polynomials for some value of x .
5. x - and y - intercepts.

NEW CONCEPTS LEARNED IN THIS LESSON INCLUDE:

- Fundamental Theorem of Algebra
- Turning Points of a Graph
- Left and Right Side Behavior of a Graph
- Sketching the Graph

THE FUNDAMENTAL THEOREM OF ALGEBRA states that every polynomial of degree 1 or more has at least one complex zero.

- If the zero is real, the graph of the polynomial will have an x -intercept.
- If the zero is non-real or complex, the graph of the polynomial will not have an x -intercept

TURNING POINT is the point where a graph changes direction from increasing to decreasing or from decreasing to increasing.

- The maximum number of turning points of the graph of a polynomial is $n - 1$ where n is the degree of the polynomial.

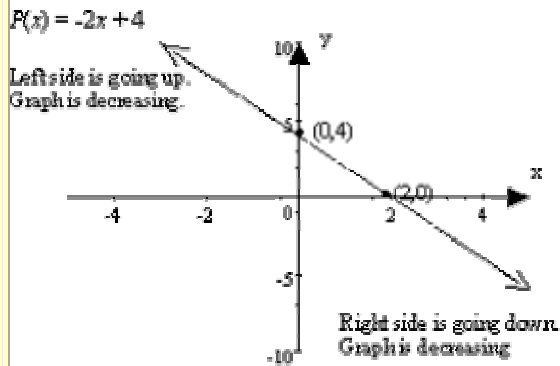
LET'S EXAMINE SOME GRAPHS.

The degree of a polynomial function and its leading coefficient determine the left side and right side behavior of the graph.

You can easily identify the graph of polynomial functions through simple observations.

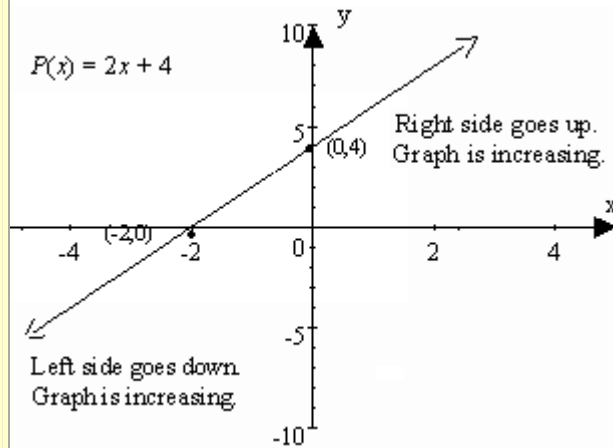
The graph of $P(x) = -2x + 4$ has:

- degree (n) = 1 and 1 is an odd number
- graph goes in opposite directions when degree is odd
- degree = 1
 - $P(x) = 0$ has 1 zero
 - the zero is a real number, so the graph has 1 x -intercept
- x -intercept (2,0)
- y -intercept (0,4)
- 0 turning points ($n - 1$)
- leading coefficient is negative



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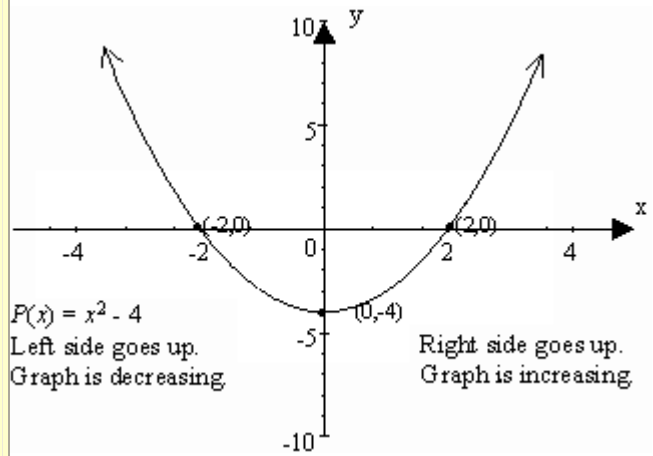


zero is a real number, so the graph has 1 x-intercept

- x-intercept (-2,0)
- y-intercept (0,4)
- 0 turning points (n - 1)
- leading coefficient is positive

The graph of $P(x) = x^2 - 4$ has:

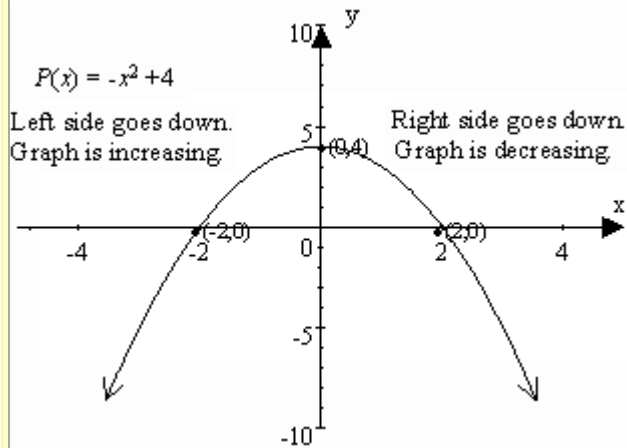
- degree (n) = 2 and 2 is an even number
- graph goes in the same direction when degree is even
- degree = 2
 - $P(x) = 0$ has 2 zeros
 - the zeros are real numbers, so the graph has 2 x-intercepts
- x-intercepts (-2,0) and (2,0)
- y-intercept (0,-4)
- 1 turning points (n - 1)



- leading coefficient is positive

The graph of $P(x) = -x^2 + 4$ has:

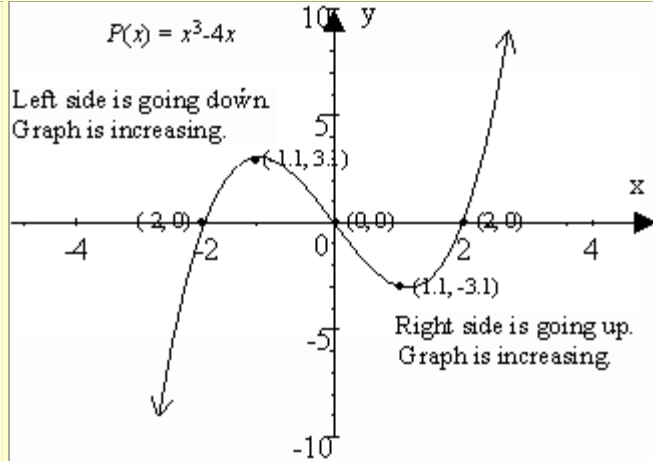
- degree (n) = 2 and 2 is an even number
- graph goes in the same direction when degree is even
- degree = 2
 - $P(x) = 0$ has 2 zeros
 - the zeros are real numbers, so the graph has 2 x-intercepts
- x-intercepts (-2,0) and (2,0)
- y-intercept (0,4)
- 1 turning points ($n - 1$)
- leading coefficient is negative



The graph of $P(x) = x^3 - 4x$ has:

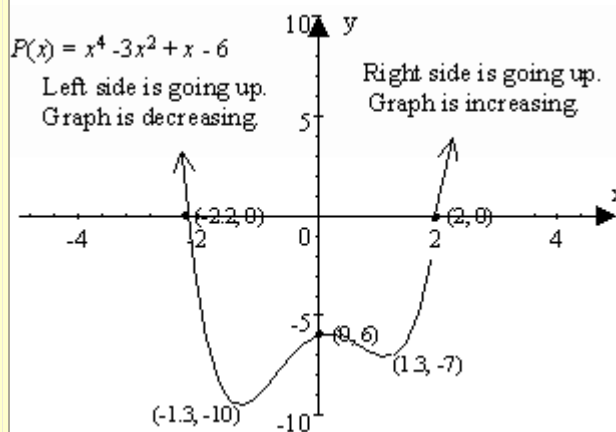
- degree (n) = 3 and 3 is an odd number
- graph goes in opposite directions when degree is odd
- degree = 3

- $P(x) = 0$ has 3 zeros the zeros are real numbers, so the graph has 3 x-intercepts
- x-intercepts $(-2,0)$, $(0,0)$ and $(2,0)$
- y-intercept $(0,0)$
- 2 turning points $(n - 1)$
- leading coefficient is positive



The graph of $P(x) = x^4 - 3x^2 + x - 6$ has:

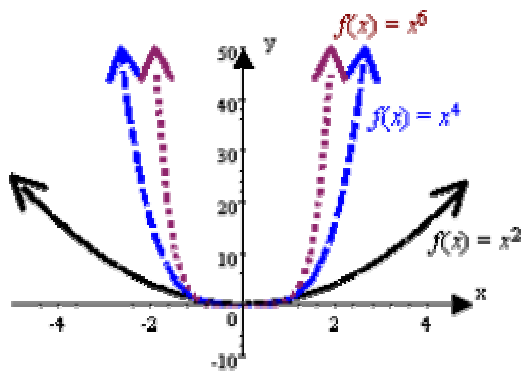
- degree $(n) = 4$ and 4 is an even number
- graph goes in the same direction when degree is even
- degree = 4
 - $P(x) = 0$ has 4 zeros
 - 2 zeros are real numbers, so the graph has 2 x-intercepts



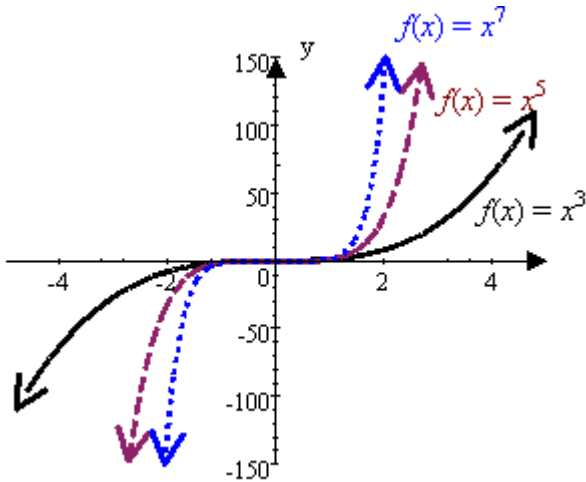
<ul style="list-style-type: none"> ○ 2 zeros are non-real • x-intercepts (-2,0) and (2,0) • y-intercept (0,-6) • 3 turning points (n - 1) • leading coefficient is positive 	
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Now, let's examine the family of graphs that have degrees that are odd or even.

Let's begin with the simplest form.



$P(x) = x^n$ where n is a positive even integer.
This graph is called a parabola.

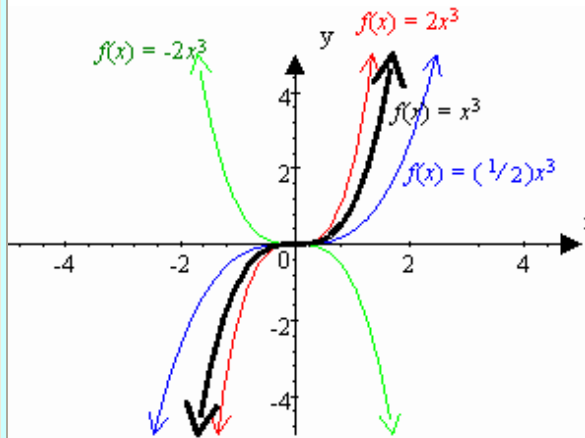


$P(x) = x^n$ where n is a positive odd integer.
This graph is called a cubic curve.

Now, let's examine the effects of the graph when the leading coefficient changes.

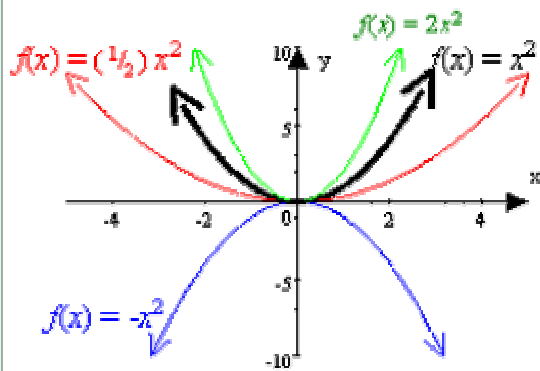
Notice how a , the leading coefficient, affects $f(x) = x^3$.

- A negative leading coefficient [reflects](#) the graph about the x -axis.
- A coefficient a such that $0 < |a| < 1$ makes the graph wider than that of $f(x) = x^3$. This is called a vertical shrink of the graph of $f(x)$.
- A coefficient a such that $|a| > 1$ makes the graph more narrow than that of $f(x) = x^3$. This is called a vertical stretch of $f(x)$.



Notice how a , the leading coefficient, affects $f(x) = x^2$.

- A negative leading coefficient reflects the graph about the x -axis.
- A coefficient a such that $0 < |a| < 1$ makes the graph wider than that of $f(x) = x^2$.
- A coefficient a such that $|a| > 1$ makes the graph more narrow than that of $f(x) = x^2$.



And what about translations of the graph of $f(x) = x^n$.

Let's examine vertical and horizontal shifts.

- A **horizontal shift** is a move along the x -axis. The graph is translated or moved left or right from the origin.
- A **vertical shift** is a move along the y -axis. The graph is translated or moved up or down from the origin.

A horizontal shift (a translation left or right) is represented by the letter h when the function is written in the form $f(x) = (x - h)^n$. To determine the value of h set $(x - h) = 0$ and solve for x .

EXAMPLE 1	EXAMPLE 2	EXAMPLE 3
$f(x) = (x - h)^n$	$f(x) = (x - 3)^3$	$f(x) = (x + 3)^4$
$x - h = 0$	$x - 3 = 0$	$x + 3 = 0$
$x = h$ and $h = x$	$x = 3$ thus, $h = 3$	$x = -3$ thus, $h = -3$

A vertical shift (a translation up or down) is represented by the letter k when the function is written in the form $f(x) = (x - h)^n + k$. The value of k is equal to the constant.

EXAMPLE 1	EXAMPLE 2	EXAMPLE 3
$f(x) = (x - h)^n + k$	$f(x) = (x - 3)^3 + 8$	$f(x) = (x + 3)^4 - 1$
$x = h, \quad y = k$	$x = 3, \quad y = 8$ thus, $h = 3, \quad k = 8$	$x = -3, \quad y = -1$ thus, $h = -3, \quad k = -1$

horizontal shift equal h units from the origin. vertical shift equal k units from the origin.	horizontal shift equal 3 units to the right from the origin. vertical shift equal 8 units up from the origin.	horizontal shift equal 3 units to the left from the origin. vertical shift equal 1 unit down from the origin.
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Observe the effects of a translation (vertical and horizontal shift) on $f(x) = x^2$ and $f(x) = x^3$.

Notice the effects of the translations (vertical and horizontal shifts) on $f(x) = x^2$.
The turning point of the parabola is called a vertex expressed as (h,k) .

$f(x) = x^2$ can be renamed as $f(x) = (x - 0)^2 + 0$ where $h = 0$ and $k = 0$.

Thus, the vertex of $f(x) = x^2$ is $(0,0)$.

The vertex of $f(x) = (x + 5)^2$ is $(-5, 0)$.

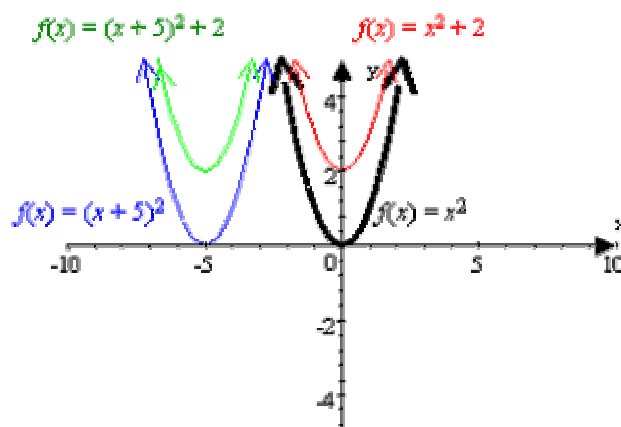
- The graph is translated 5 units to the left and
- 0 units up/down as compared to the graph of $f(x) = x^2$.

The vertex of $f(x) = (x + 5)^2 + 2$ is $(-5, 2)$.

- The graph is translated 5 units to the left and
- 2 units up as compared to the graph of $f(x) = x^2$.

The vertex of $f(x) = x^2 + 2$ is $(0, 2)$.

- The graph is translated 0 units left/right and
- 2 units up as compared to the graph of $f(x) = x^2$.



Notice the effects of the translations (vertical and horizontal shifts) on $f(x) = x^3$.

$f(x) = x^3$ can be renamed as $f(x) = (x - 0)^3 + 0$ where $h = 0$ and $k = 0$.

Thus, $f(x) = x^3$ has (h, k) at $(0,0)$.

The (h, k) of $f(x) = (x - 5)^3$ is $(5, 0)$.

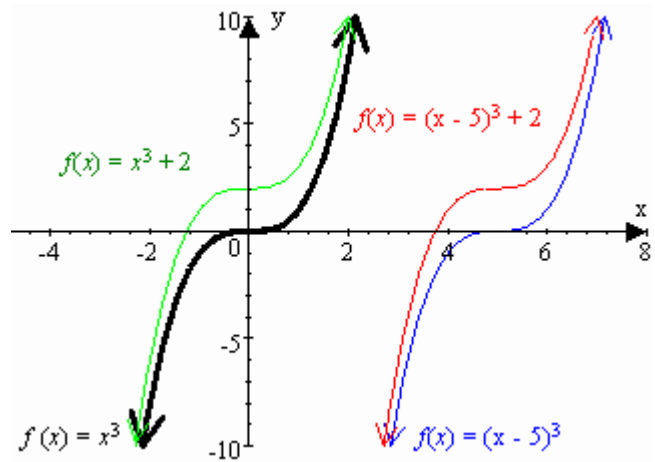
- The graph is translated 5 units to the right and
- 0 units up/down as compared to the graph of $f(x) = x^3$.

The (h, k) of $f(x) = x^3 + 2$ is $(0, 2)$.

- The graph is translated 0 units to the left/right and
- 2 units up as compared to the graph of $f(x) = x^3$.

The (h, k) of $f(x) = (x - 5)^3 + 2$ is $(5, 2)$.

- The graph is translated 5 units to the right and
- 2 units up as compared to the graph of $f(x) = x^3$.



In this lesson you will graph polynomial functions using its zeros.

Let's look at some examples.

EXAMPLE 1: $f(x) = x(x - 2)(x + 3)$

Let's first analyze what we know.

- The polynomial function is factored completely.
- The polynomial function has 3 linear factors.
- Thus, the polynomial function has 3 distinct zeros.
- Apply the Zero-Factor Property to find the zeros.
- Zeros are $x = 0$; $x = 2$; $x = -3$, and
- corresponding x -intercepts are $(0,0)$; $(2,0)$; $(-3,0)$.

We also know the following about the polynomial function.

1. Degree = 3
2. The degree of 3 is an odd number.
3. The leading coefficient is positive.

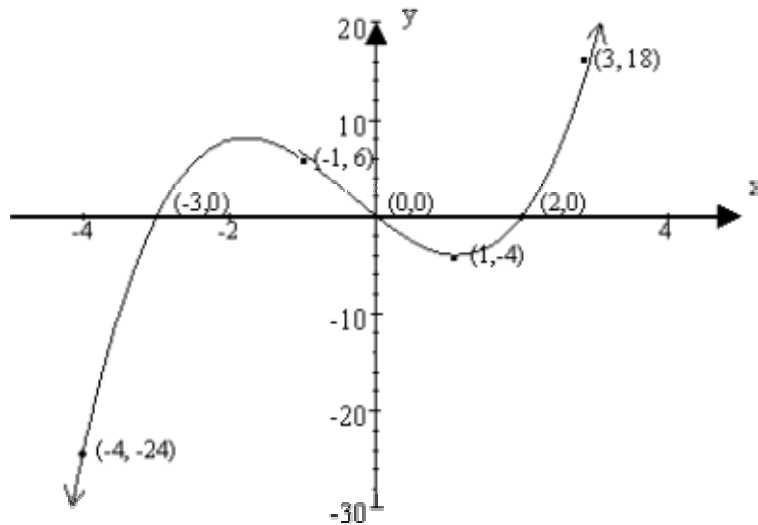
4. The left side of the graph goes down.
5. The right side of the graph goes up.
6. The graph has 2 turning points.

To complete the graph of the polynomial function we need to know the curvature of the graph on each side of the x -intercepts. Finding points from each interval will complete the sketch and then we can graph a more accurate curve.

x	-4	-1	1	3
$f(x) =$	$f(-4) = -$	$f(-1) = -$	$f(1) =$	$f(3) =$
$x(x - 2)(x + 3)$	$4(-4 - 2)(-4 + 3)$	$1(-1 - 2)(-1 + 3)$	$1(1 - 2)(1 + 3)$	$3(3 - 2)(3 + 3)$
$f(x) =$	$f(-4) = -24$	$f(-1) = 6$	$f(1) = -4$	$f(3) = 18$

Plot the x -intercepts and the points just found on the graph and draw a smooth curve passing through all these points.

$$f(x) = x(x - 2)(x + 3)$$



EXAMPLE 2: $f(x) = x^4 - 4x^2$

Again, let's first analyze what we know.

- The polynomial function needs to be factored completely.
 - $f(x) = x^4 - 4x^2$
 - $f(x) = x^2(x^2 - 4)$
 - $f(x) = x^2(x - 2)(x + 2)$
- The polynomial function has 1 quadratic and 2 linear factors.
- Thus, the polynomial function has 4 distinct zeros.

- Apply the Zero-Factor Property to find the zeros.
- Zeros are $x = 0$ of *multiplicity 2; $x = -2$; $x = 2$, and
- corresponding x-intercepts are $(0,0)$; $(-2,0)$; $(2,0)$.

* When the multiplicity of a zero is even, the curve has a turning point at that zero.

When the multiplicity of a zero is odd, the curve passes through that zero.

We also know the following about the polynomial function.

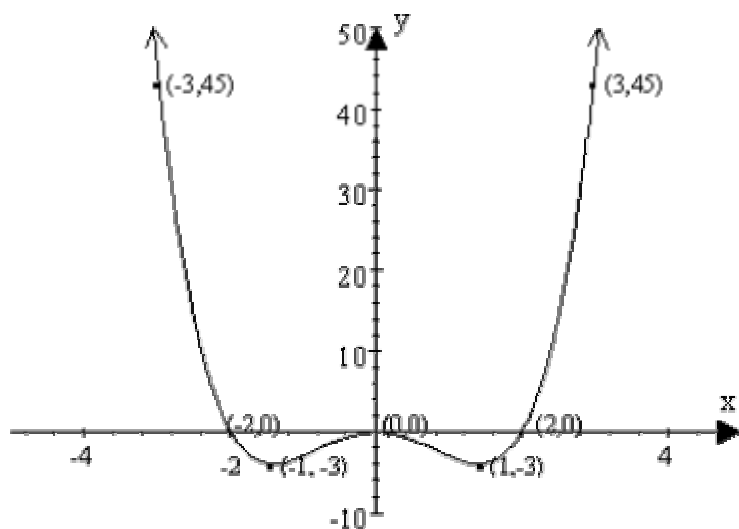
1. Degree = 4
2. The degree of 4 is an even number.
3. The leading coefficient is positive.
4. The left side of the graph goes up.
5. The right side of the graph goes up.
6. The graph has 3 turning points.

To complete the graph of the polynomial function we need to know the curvature of the graph on each side of the x-intercepts.

Finding points from each interval will complete the sketch and then we can graph a more accurate curve.

x	-3	-1	1	3
$f(x) = x^4 - 4x^2$	$f(-3) = (-3)^4 - 4(-3)^2$	$f(-1) = (-1)^4 - 4(-1)^2$	$f(1) = 1^4 - 4(1)^2$	$f(3) = 3^4 - 4(3)^2$
$f(x) =$	$f(-3) = 45$	$f(-1) = -3$	$f(1) = -3$	$f(3) = 45$

Plot the x-intercepts and the points just found on the graph and draw a smooth curve passing through all these points.



Notice the turning point at $(0,0)$. $x = 0$ has multiplicity of 2.

Using technology to graph polynomial functions. A TI-82 graphing calculator will be used to illustrate some features that will graph polynomial and other functions.

PRELIMINARIES:

1. Keying in functions
 1. Remember, all equations must be in the form $y =$ some algebraic expression.
 2. Right below the display window is the $Y=$ key. You can enter up to 10 equations.
 3. Use arrows to scroll up or down and left or right.
 4. An equation is active when the $=$ is blocked.
Activation and de-activation of equations can be done by using back arrow to block the equal sign and enter. The $=$ sign is a toggle key which will reverse the command when the enter key is pressed. (i.e., If "off", the command will turn "on" when you block $=$ and press enter and vise-versa..)
 5. Use the $X,T,0$ key to enter a variable x .
 6. Use parenthesis to group numerators, denominators, and exponents that are not monomials. Also, use parenthesis around coefficients that are fractions.
 7. Use the $^$ key to enter exponents.
2. Graphing functions.
 1. Once you have keyed in an equation, press the GRAPH key. All active equations will be graphed.
 2. Equations are graphed either sequential or

- simultaneous and that can be changed by pressing MODE key.
3. Once in the MODE command, a blinking word indicates which command is active. Using the arrow keys you can activate a different command.
 4. By pressing QUIT (2nd MODE) you can terminate a command and return to the display window.
3. Viewing a graph.
1. Often times the displayed graph is not the most complete picture. Thus, it is necessary to adjust the viewing window.
 2. The viewing window is a rectangular section of the coordinate plane.
 3. Adjustments can be made by pressing the WINDOW key. This command will allow you to enter the range of the viewing window. Enter the coordinates of the x-axis. A minimum value for x (Xmin), a maximum value (Xmax) and the scaling increment unit (Xscl). The same goes for the y-axis. Enter the Ymin, Ymax, and Yscl.
 4. You can also use the ZOOM key to adjust the viewing window. You have many options in the zoom menu. #2 will zoom in which will decrease the range of the viewing window and #3 zoom out will increase the range. The blinking asterisk * or cursor can be moved by using the arrows to a target point from which to zoom from.
4. Resetting your graphing calculator to its standard features.
1. Press MEM (2nd +)
 2. Enter a 3 or scroll down to RESET.
 3. Enter a 2 or scroll down to RESET.
 4. If a blank screen appears, press 2nd and the up arrow until the display is visible. You should see "Mem cleared".

Use of the graphing calculator should complement your ability to graph using pencil and paper. You can examine a graph by pressing the TRACE key and moving the cursor left or right. The coordinates of each point can be seen at the bottom of the screen.

You can also view these coordinates in table format by pressing TABLE (2nd GRAPH). You can adjust the values of the table by pressing TblSet ((2nd WINDOW).

In the TblSet options, TblMin = the value of the first x-value displayed in the table and Delta Tbl = the increment value of x. The

increment in decimal units is useful in approximating zeros and turning points more precisely.

Let's practice graphing a few polynomials with the use of a graphing calculator.

EXAMPLE 1: $f(x) = x^4 + 8x^3 - 2x^2 + 7x - 10$

Knowledge that you already have of this polynomial should include the following:

- 4 distinct zeros
- 3 turning points
- Left side goes up and Right side goes up
- y-intercept is (0, -10)

So, the following is the sequence of key strokes for a TI-82 graphing calculator. Each key stroke is separated by a comma.

Y=, XT0, ^, 4, +, 8, XT0, ^, 3, -, 2, XT0, x2, +, 7, XT0, -, 10, GRAPH

If the viewing window is on standard setting, the complete turning point of the graph is not displayed (See figure (a) below). The range of the viewing window needs adjustment. The following range will display a more complete picture (See figure (b) below).

WINDOW

Xmin = -10

Xmax = 10

Xscl = 1

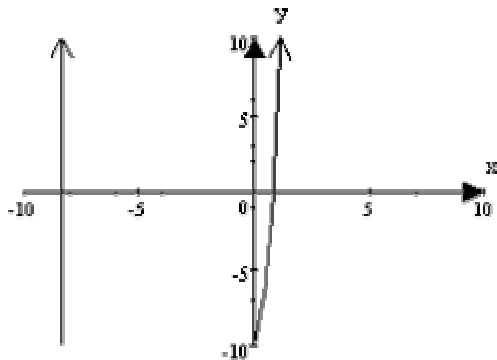
Ymin = -600

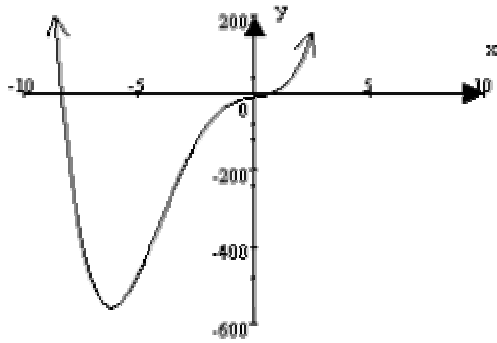
Ymax = 200

Yscl = 100

Compare the two graphs.

Figure (a) Figure (b)





EXAMPLE 2: Exercise 69 from page 301 of the textbook, *College Algebra*, 4th Edition by Larson and Hostetler.

Advertising Costs A company that produces portable cassette players estimates that the profit for selling a particular model is given by

$$P = -76x^3 + 4830x^2 - 320,000 \quad 0 \leq x \leq 60$$

where P is the profit in dollars and x is the advertising expense in tens of thousands of dollars. Using this model, find the smaller of two advertising amounts that yields a profit of \$2,500,000.

Knowledge that you already have of this polynomial should include the following:

- 3 distinct zeros
- 2 turning points
- Left side goes up and Right side goes down
- y-intercept is (0, -320,000)

So, the following is the sequence of key strokes for a TI-82 graphing calculator. Each key stroke is separated by a comma.

Y=, (-), 76, XT0, ^, 3, +, 4830, XT0, ^, 2, - 320000, GRAPH

If the viewing window is on standard setting, the complete turning point of the graph is not displayed (See figure (a) below). The range of the viewing window needs adjustment. The following range will display a more complete picture (See figure (b) below).

WINDOW

Xmin = 0

Xmax = 60

Xscl = 10

Ymin = -500000

$$Y_{\max} = 5000000$$

$$Y_{\text{scl}} = 10000$$

Compare the two graphs.

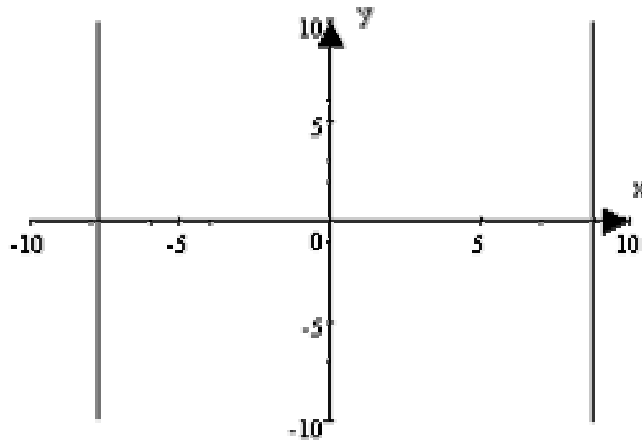
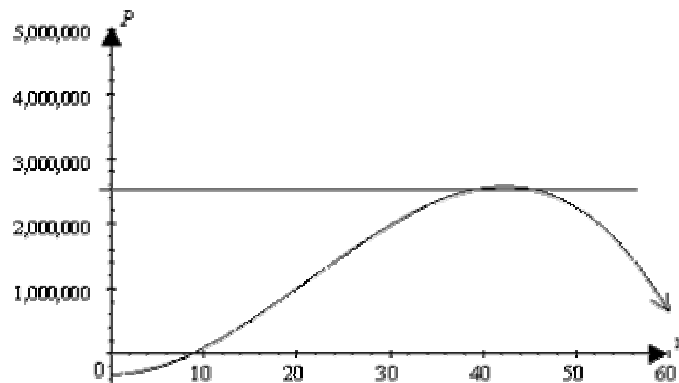


Figure (a)

Figure (b)



A horizontal line at $P = 2,500,000$ has been drawn in Figure (b) to focus your attention where the two advertising amounts appear on the graph.

Now use the TRACE or TABLE to approximate the smaller of these two amounts that yields a profit of \$2,500,000.

Answer: Approximately 38 ten thousand (\$38,000) dollars yields a profit of about \$2,500,000.

The other advertising amount is approximately 45 ten thousand (\$45,000) dollars. The smaller, however, is 38.

