

2.6. Rational zeros of polynomial functions.

In this lesson you will learn to find zeros of polynomial functions that are not factorable.

REVIEW OF PREREQUISITE CONCEPTS:

- A polynomial of n^{th} degree has precisely n distinct zeros.
- Every polynomial function of odd degree having real coefficients has at least one real zero.
- Real zeros are the x -intercepts of the graph of a function and non-real zeros do not lie on the x -axis.
- Non-real zeros are conjugates so they always occur in pairs.

There exists some theorems that are useful in determining zeros of polynomial functions.

- Descartes' Rule of Signs is useful in determining the number of possible positive and negative real zeros and possible non-real zeros.
- Rational Zeros Theorem is useful in determining the set of possible rational zeros of a polynomial function with integer coefficients.
- Boundness Theorem is useful in determining if any real number is greater than or less than all real zeros of a polynomial function with real coefficients.

Each theorem is discussed further in the following pages.

Descartes' Rule of Signs

Let $P(x)$ be a polynomial with real coefficients and terms in descending powers of x

(a) The number of positive real zeros of $P(x)$ either equals the number of *variations* in sign occurring in the coefficients of $P(x)$, or is less than the number of variations by a positive even integer.

(b) The number of positive real zeros of $P(-x)$ either equals the number of variations in sign occurring in the coefficients of $P(-x)$, or is less than the number of variations by a positive even integer.

Each time the number of variations is decreased by a positive even integer, the number of possible non-real zeros increases by the same positive even integer because non-real zeros occur in pairs.

EXAMPLE: $P(x) = 2x^4 + 7x^3 - 17x^2 - 58x - 24$

We already know the following about $P(x)$.

- Degree = 4, so the polynomial function has 4 distinct zeros.
- The left and right side of the graph both go up.
- The graph has 3 turning points.
- The y-intercept is (0, -24).

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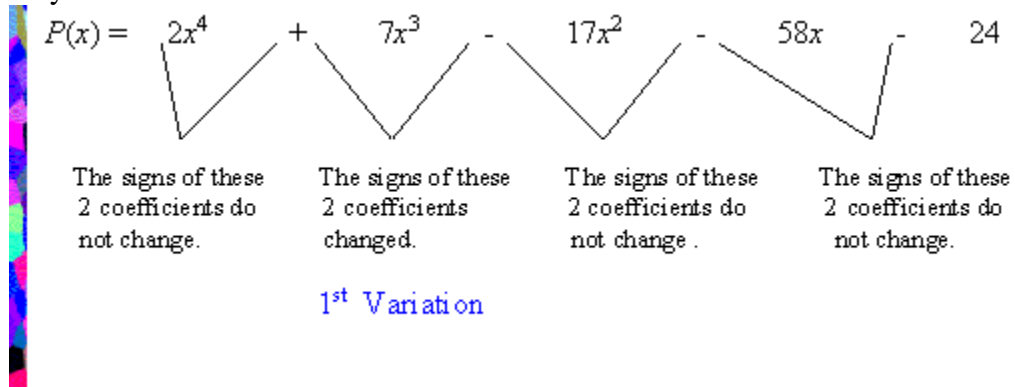
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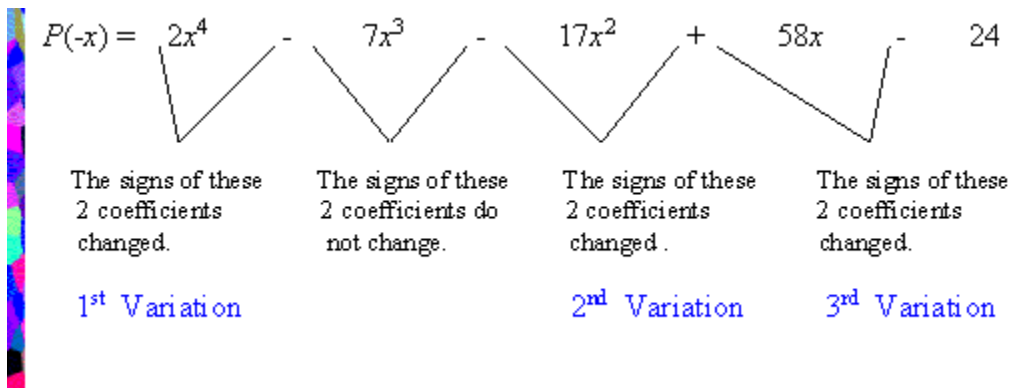
Because there is only 1 variation for the polynomial function when x is positive, there is only 1 positive real zero.

Now, let's look at how the signs of consecutive integers vary for $P(-x)$.

Substitute all x 's with $(-x)$ and simplify the polynomial.

$$P(-x) = 2(-x)^4 + 7(-x)^3 - 17(-x)^2 - 58(-x) - 24$$

$$P(-x) = 2x^4 - 7x^3 - 17x^2 + 58x - 24$$



Because there are 3 variations for the polynomial function when x is negative, there are 3 negative real zeros or 1 (the number of variations decreased by a positive even integer which is $3 - 2$).

Descartes' Rule of Signs can be illustrated in table form.

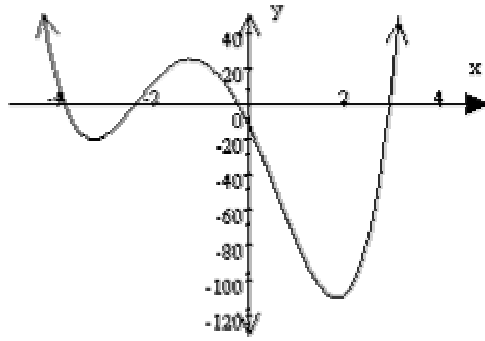
We know that the total number of distinct zeros has to equal the number of positive real zeros plus the number of negative real zeros plus the non-real zeros.

Distinct zeros = Positive real zeros + Negative real zeros + *Non-real zeros
 For the polynomial function $P(x) = 2x^4 + 7x^3 - 17x^2 - 58x - 24$ the following is true.

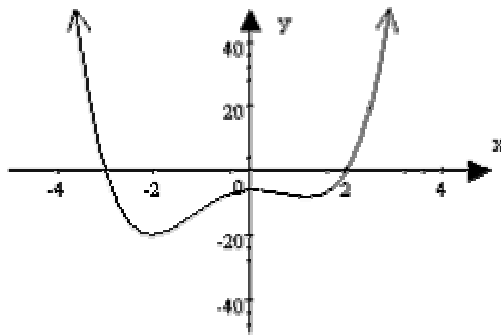
Total Distinct Zeros	Number of Positive real zeros	Number of Negative real zeros	*Number of non-real zeros
4	1	3	0
4	1	1	2

* Remember, when the number of positive or negative real zeros is decreased by a positive even integer the non-reals are increased by that same number.

The following sketches are possible graphs of $P(x) = 2x^4 + 7x^3 - 17x^2 - 58x - 24$ based on Descartes' rule of signs.



This graph has 3 negative and 1 positive real zeros



This graph has 1 negative and 1 positive real zeros
and 2 non-real zeros

Rational Zeros Theorem

Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ where $a_n \neq 0$, be a polynomial with integer coefficients.

If $\frac{p}{q}$ is a rational number written in lowest terms and if $\frac{p}{q}$ is a zero of $P(x)$, then p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n .

EXAMPLE: $P(x) = 2x^4 + 7x^3 - 17x^2 - 58x - 24$

We already know the following about this polynomial function.

- Degree = 4, so the polynomial function has 4 distinct zeros.
- The left and right side of the graph both go up.
- The graph has 3 turning points.
- The y-intercept is (0, -24).
- It is possible for the graph to have either 3 negative and 1 positive real zeros or
- 1 negative and 1 positive real zeros and 2 non-real zeros.

Let's apply the Rational Zeros Theorem to find the set of **possible** rational zeros.

We know that p is a factor of the constant term a_0 ($a_0 = 24$) and q is a factor of the leading coefficient a_n ($a_n = 2$).

$$\frac{p}{q} = \frac{\text{factors of the constant term}}{\text{factors of the leading coefficient}} = \frac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24}{\pm 1, \pm 2}$$

The following is a list of all the rational numbers that were formed from the factors of p and q .

$$\pm \frac{1}{1}, \pm \frac{2}{1}, \pm \frac{3}{1}, \pm \frac{4}{1}, \pm \frac{6}{1}, \pm \frac{8}{1}, \pm \frac{12}{1}, \pm \frac{24}{1}, \pm \frac{1}{2}, \pm \frac{2}{2}, \pm \frac{3}{2}, \pm \frac{4}{2}, \pm \frac{6}{2}, \pm \frac{8}{2}, \pm \frac{12}{2}, \pm \frac{24}{2}$$

However, these rational numbers written in lowest terms reduces the list of possible rational zeros to

$$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24, \pm \frac{1}{2}, \pm \frac{3}{2}$$

Thus any rational zero of $P(x)$ will come from this list.

The numbers from the list can be checked using synthetic division to verify if any are rational zeros of $P(x)$.

However, before we check any numbers, it is useful to know about the next theorem.

Boundness Theorem

Let $P(x)$ be a polynomial with real coefficients. If $P(x)$ is divided synthetically by $x - c$, and

(a) if $c > 0$ and all numbers in the bottom row of the synthetic division are all positive or all negative (with 0 considered positive or negative, as needed), then $P(x)$ has no real zero greater than c .

c is said to be an upper bound.

(b) if $c < 0$ and the numbers in the bottom row of the synthetic division alternate in sign (with 0 considered positive or negative, as needed), then $P(x)$ has no zeros less than c .

c is said to be a lower bound.

The Boundness theorem is useful in eliminating those numbers from the list that are less than or greater than all real zeros of the polynomial function.

So, as we do synthetic division, we must keep watch of the signs of the bottom row.

$$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24, \pm \frac{1}{2}, \pm \frac{3}{2}$$

$$\begin{array}{r|rrrrr} 1) & 2 & 7 & -17 & -58 & -24 \\ & & 2 & 9 & -8 & -66 \\ \hline & 2 & 9 & -8 & -66 & -90 \end{array}$$

One can see that -90 is not a zero but $(1, -90)$ is a point of the graph. Check your bottom row. 1 is neither an upper bound nor a lower bound.

$$\begin{array}{r|rrrrr} -1) & 2 & 7 & -17 & -58 & -24 \\ & & -2 & -5 & 22 & 36 \\ \hline & 2 & 5 & -22 & -36 & 12 \end{array}$$

$(-1, 12)$ is a point of the graph of $P(x)$.
 -1 is neither an upper bound nor a lower bound.

$$\begin{array}{r|rrrrr} 2) & 2 & 7 & -17 & -58 & -24 \\ & & 4 & 22 & 10 & -96 \\ \hline & 2 & 11 & 5 & -48 & -120 \end{array}$$

$(2, -120)$ is a point of the graph of $P(x)$.
 2 is neither an upper bound nor a lower bound.

$$\begin{array}{r|rrrrr} -2) & 2 & 7 & -17 & -58 & -24 \\ & & -4 & -6 & 46 & 24 \\ \hline & 2 & 3 & -23 & -12 & 0 \end{array}$$

WOW! A ZERO!!!
 -2 is neither an upper bound nor a lower bound.

Having found the first zero, apply the Division Algorithm to write the polynomial function in factored form.

$$P(x) = 2x^4 + 7x^3 - 17x^2 - 58x - 24$$

$$P(x) = (x + 2)(2x^3 + 3x^2 - 23x - 12)$$

We can now focus on factoring the cubic polynomial. The list of possible rational zeros for $P(x)$ can still be used to find remaining zeros.

It is possible that any zero may have multiplicity, so each zero should be checked a second time using the depressed (quotient) polynomial.

$$P(x) = (x+2)(2x^3 + 3x^2 - 23x - 12)$$

$$\begin{array}{r} -2) \ 2 \quad 3 \quad -23 \quad -12 \\ \underline{\quad -4 \quad 2 \quad 42} \\ 2 \quad -1 \quad -21 \quad 30 \end{array} \quad -2 \text{ does not have multiplicity.}$$

$$\begin{array}{r} 3) \ 2 \quad 3 \quad -23 \quad -12 \\ \underline{\quad 6 \quad 27 \quad 12} \\ 2 \quad 9 \quad 4 \quad 0 \end{array} \quad \text{WOW ANOTHER ZERO!!}$$

Notice, 3 is an upper bound. That means 4, 6, 8, 12, and 24 are numbers greater than all real zeros of $P(x)$ and can be eliminated

The polynomial function in factored form can now be written as $P(x) = (x+2)(x-3)(2x^2 + 9x + 4)$.

NOTICE: The factors are now linear and quadratic. You can use any of the following methods to find the remaining zeros of the quadratic polynomial.

1. Factor the polynomial
2. Apply the square root property
3. Apply the quadratic formula
4. Complete the square
5. Continue the trial and error method using synthetic division.

Let's see. The polynomial function factors completely as $P(x) = (x+2)(x-3)(2x+1)(x+4)$,

Thus, the zeros of $P(x)$ are -2, 3, $-1/2$, and -4.

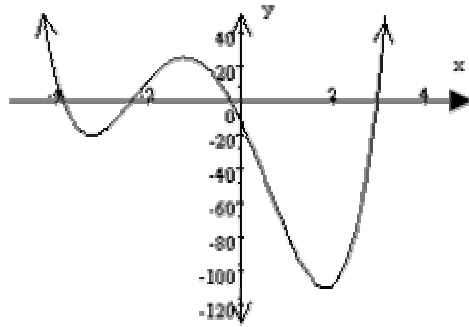


LOOK!!! 3 of the zeros are negative and 1 zero is positive.

Using the following information, we can now graph $P(x) = 2x^4 + 7x^3 - 17x^2 - 58x - 24$.

- Zeros: $x = -2$, $x = 3$, $x = -1/2$, and $x = -4$.
- Additional points found while doing synthetic division: (1, -90); (-1, 12); and (2, -120)

- Find $P(-3)$ to learn what the graph does in the interval $[-2, -4]$. $P(-3) = -30$



$$P(x) = 2x^4 + 7x^3 - 17x^2 - 58x - 24$$