2.7. Irrational zeros of polynomial functions.

You have learned to find zeros of polynomials that factor and polynomials of higher degree that have rational zeros.

In this lesson you will learn to find approximations of irrational zeros.

Irrational zeros, like complex zeros, also occur in pairs called conjugates.

A new theorem helpful in approximating zeros is the Intermediate Value Theorem.

INTERMEDIATE VALUE THEOREM

Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a)$ and $f(b)$ are opposite in sign,
then there exists at least one zero in the interval $[a, b]$.

Before we apply the Intermediate Value Theorem, it is necessary to learn a shortcut to synthetic division.

This shortcut was not presented earlier because too many other concepts were being introduced. In the earlier lessons, each synthetic division was performed as a single problem.

The shortcut entails that the row of products (middle row) be eliminated and only the bottom row be displayed.

That means that the multiplication and vertical sums are calculated mentally.

EXAMPLE: Compare the shortcut procedure illustrated in one table with the two separate division problems.

SHORTCUT 2 SEPARATE DIVISION PROBLEMS

	$P(x) = 2x^3 + 3x^2 - 23x - 12$
$P(x)$	2 3 -23 -12
-2	2 -1 -21 30
3	2 9 4 0

$P(x) = (x + 2)(2x^3 + 3x^2 - 23x - 12)$

$$
\begin{array}{r|rrrr}
-2) & 2 & 3 & -23 & -12 \\
 & & -4 & 2 & 42 \\
\hline
 & 2 & -1 & -21 & 30
\end{array}
$$

-2 does not have multiplicity.

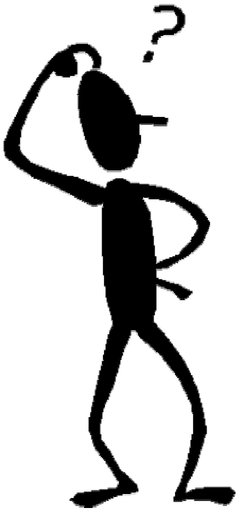
$$
\begin{array}{r|rrrr}
3) & 2 & 3 & -23 & -12 \\
 & & 6 & 27 & 12 \\
\hline
 & 2 & 9 & 4 & 0
\end{array}
$$

WOW ANOTHER ZERO!!

Notice, 3 is an upper bound. That means 4, 6, 8, 12, and 24 are numbers greater than all real zeros of $P(x)$ and can be eliminated

In the shortcut method, the top row is the dividend and each subsequent row is the bottom row of each single division problem.

If you have difficulty understanding the shortcut, you may want to perform each synthetic division separately and try to connect the two ideas.



Verify if any of the following numbers are zeros of $P(x) = 6x^3 - 17x^2 - 15x + 36$.

$x = -3$, $x = -1$, $x = 2$, $x = 3$, $x = 5$

$P(x)$	$P(x) = 6x^3 - 17x^2 - 15x + 36$	COMMENTS
	6 -17 -15 36	(0, 36) is a *y*-intercept
-3	6 -35 90 -234	(-3, -234) a point of the graph and **-3 is a lower bound**
-1	6 -23 8 28	(-1, 28) a point of the graph
2	6 -5 -25 -14	(2, -14) a point of the graph
3	6 1 -12 0	(3, 0) **WOW! A ZERO....ALRIGHT..**
*5	6 13 50 286	(5, 286) a point of the graph and **5 is an upper bound**

* There is no need to continue dividing into the top row since 3 is a zero. You have two options, (1) you can start a new table using the quadratic polynomial as the first row or
(2) solve the quadratic polynomial by any other method.

Given $P(x) = 6x^3 - 17x^2 - 15x + 36$, we found $x = 3$ to be a zero of $P(x)$.

So, in factored form $P(x) = (x - 3)(6x^2 + x - 12)$

Guess What? Having solved the quadratic equation $6x^2 + x - 12 = 0$, we found the other two zeros to be $(-\,{}^{3}/_{2})$ and $({}^{4}/_{3})$.

$(-\,{}^{3}/_{2})$ is in the interval [-3, -1] and

$({}^{4}/_{3})$ is in the interval [0, 2].

Let's try an example that has no rational zeros and the polynomial does not factor.

$P(x) = -x^4 - 6x^3 + 5x^2 + 2x - 1$

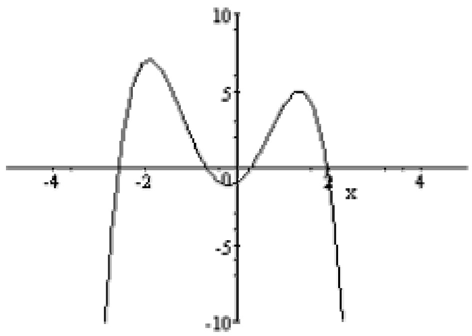
First, let's analyze what we already know about the polynomial.

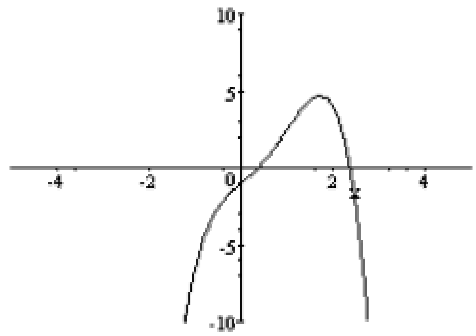
- Degree = 4 so the polynomial function has 4 distinct zeros.
- The graph of $P(x)$ goes down on both left and right sides.
- The graph of $P(x)$ has 3 turning points.
- (0, -1) is the y-intercept.
- The possible rational zeros are $\pm$ 1.
- The number of possible positive and negative zeros is as follows:

Total distinct zeros	Total Positive Real Zeros	Total Negative Real Zeros	Total Non-Real Zeros
4	2	2	0
4	2	0	2
4	0	2	2
4	0	0	4

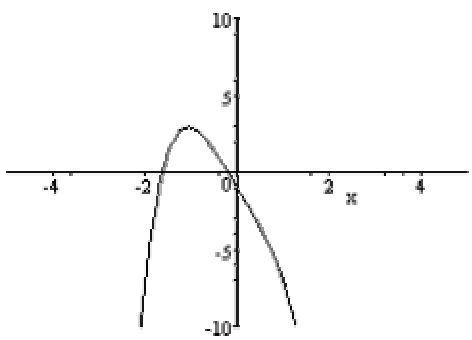
- Each row of the table above serves to sketch a possible graph of $P(x)$.

2 positive and 2 negative real zeros 0 non-real zeros	2 positive and 0 negative real zeros 2 non-real zeros

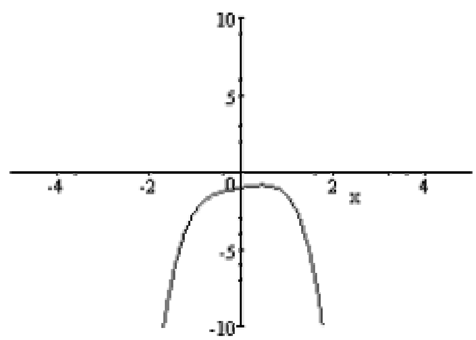




0 positive and 2 negative real zeros
2 non-real zeros



0 positive and 0 negative real zeros
4 non-real zeros



First we should verify if either 1 or -1 are zeros of $P(x) = -x^4 - 6x^3 + 5x^2 + 2x - 1$.

P(x)	$P(x) = -x^4 - 6x^3 + 5x^2 + 2x - 1$	Comments
	-1 - 6 5 2 -1	(0, -1) the *y*-intercept
-1	-1 -5 10 -8 7	(-1, 7) a point of the graph and there exists a zero in the interval [-1, 0]

1	-1 -7 -2 0 -1	(1, -1) a point of the graph and **1 is an upper bound** **-- all signs of the bottom row are negative**

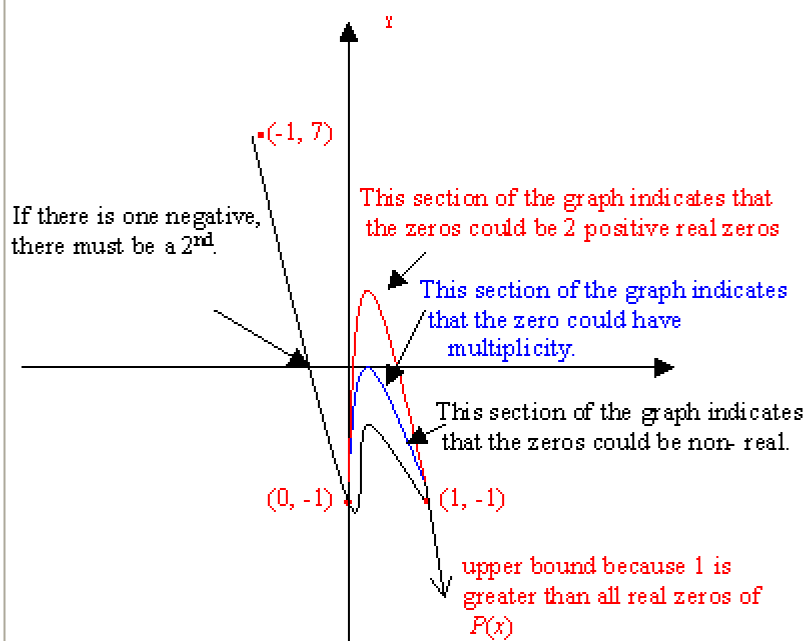
NOW, let's analyze all the information gathered from the table above.

There exists a zero in the interval [-1,0] because the *y*-values of the points (-1,7) and (0, -1) are opposite signs.

If there is one negative, there must be a second because of Descartes Rule of Signs.

If there are 2 negatives then the other 2 zeros are either positives or non-real.

The zero(s) must be in the interval [0,1] because 1 is an upper bound. There are no more zeros greater than 1.



With the use of a graphing calculator, the zeros can be approximated with greater precision.
First, we will approximate zeros manually to the nearest tenth, and then we will learn to use a graphing calculator.

Remember three of the zeros are in the intervals [-1, 0] and [0, 1]. This means that -0.9 , -0.8, -0.7, -0.6, -0.5, -0.4, -0.3, -0.2, -0.1 could be approximations of zero for [-1, 0].

To verify each of these numbers, we will use the shortcut method of synthetic division.

BUT WAIT!.. ARE YOU REALLY GOING TO TRY ALL OF THE NUMBERS? Or should we analyze the situation?

Let's see. Start at -0.5 and eliminate one-half of the zeros. Right?

P(x)	*P*(x) = $-x^4 - 6x^3 + 5x^2 + 2x - 1$	Comments
	-1 -6 5 2 -1	(0, -1) the *y*-intercept
-0.1		
-0.2		
-0.3		
-0.4		
-0.5	-1 -5.5 7.5 -1.875 -0.0625	(-0.5, -0.0625) is a point very close to the *x* axis. Because the remainder is still negative, we eliminate the numbers -0.1, -0.2, -0.3, and -0.4. We should try -0.6 to determine which is a better approximation.
-0.6		
-0.7		
-0.8		
-0.9		

P(x)	*P*(x) = $-x^4 - 6x^3 + 5x^2 + 2x - 1$	Comments
	-1 -6 5 2 -1	(0, -1) the *y*-intercept
-0.1		
-0.2		
-0.3		
-0.4		
-0.5	-1 -5.5 7.5 -1.875 -0.0625	(-0.5, -0.0625) is a point very close to the *x* axis. Because the remainder is still negative, we eliminate the numbers -0.1, -0.2, -0.3, and -0.4. We should try -0.6 to determine which is a better approximation.
-0.6	-1 -5.4 8.24	**NOPE!** Not a better approximation. The point

	-2.944 0.7664	(-0.6, 0.7664) is further away from the *x* axis.
-0.7		
-0.8		
-0.9		

So, the zero in the interval [-1, 0] approximated to the nearest tenth is $x =$ -0.5. Now, if you wanted to continue approximating to the nearest hundredth, you would verify the values in the interval [-0.5, -0.6] to two decimal places.

These numbers are: -0.51, -0.52, -0.53, -0.54, -0.55, -0.56, -0.57, -0.58, and -0.59

Let's find zeros in the second interval [0, 1]. Supposedly there are 2 zeros in this interval.

EXAMINE the order in which the steps were done.

P(x)	$P(x) = -x^4 - 6x^3 + 5x^2 + 2x - 1$	Comments
	-1 -6 5 2 -1	(0, -1) the *y*-intercept
0.1		
0.2		
0.3	-1 -6.3 3.11 2.933 -0.1201	Step 3: (0.3, -0.1201) is the best approximation of the 1st zero in the interval [0, 1]
0.4	-1 -6.4 2.44 2.976 0.1904	Step 2: (0.4, 0.1904) the point is closer to the *x* axis.
0.5	-1 -6.5 1.75 2.875 0.4375	Step 1: (0.5, 0.4375) the point is not very close to the *x* axis.
0.6	-1 -6.6 1.04 2.624 0.5744	Step 4: (0.6, 0.5744) still positive and not a good approximation.
0.7	-1 -6.7 0.31 2.217 0.5519	Step 5: (0.7, 0.5519) still positive but *y* value is decreasing.
0.8	-1 -6.8	Step 6: (0.8, 0.3184) still

	0.44 1.648 0.3184	positive and *y* value is still decreasing.
0.9	-1 -6.9 - 1.21 0.911 - 0.1801	Step 7: (0, -0.1801) is the best approximation of the 2nd zero in the interval [0, 1]

So far we have found three zeros of *P*(x) = $-x^4 - 6x^3 + 5x^2 + 2x - 1$ approximated to the nearest tenth, -0.5, 0.3, and 0.9.

Where is the fourth zero?

- The fourth zero has to be negative because we already have 2 positive real zeros and 1 negative real zero.
- The fourth zero has to be less than -1.

First, we do a trial and error approach for the integers less than -1. Then we can approximate to the nearest tenth.

P(x)	*P*(x) = $-x^4 - 6x^3 + 5x^2 + 2x - 1$	Comments
	-1 -6 5 2 -1	(0, -1) the *y*-intercept
-1	-1 -5 10 -8 7	Step 1: We have already done (-1, 7). Writing it here makes it easier to compare remainders.
-2		
-3		
-4		
-5	-1 -1 10 -48 239	Step 2: (-5, 239) the *y* value is still positive so the zero is less than -5.
-6	-1 0 5 -28 167	Step 4: (-6, 167) is getting closer to the *x* axis.
-7	-1 1 - 2 16 -113	Step 5: (-7, -113) The *y* value changed sign. So, the zero is in the interval [-7, -6].
-8	-1 2 - 11 90 -721	Step 3: (-8, -721) **OOOOPS!** The *y* value is now negative. So, the zero is between -5 and -8.
-9		

The fourth zero is in the interval [-7, -6]

Find the fourth zero approximated to the nearest tenth.

DID YOU GET (- 6.7, - 0.4841)? GOOD FOR YOU!

Now, let's try some problems.