

3.1. RATIONAL EXPRESSIONS

RATIONAL NUMBERS

In previous courses you have learned how to operate (do addition, subtraction, multiplication, and division) on rational numbers (fractions).

Rational numbers are quotients of 2 integers in which the denominator does not equal zero. If the denominator does equal zero, the rational number is said to be undefined.

Which of the following examples are undefined.

A) $\frac{5}{0}$ B) $\frac{0}{(-3)^2+1}$ C) $\frac{0}{5}$ D) $\frac{0}{4-4}$

Did you say, examples A, $\frac{5}{0}$, and example D, $\frac{0}{4-4}$, are undefined? Then examples B and C are said to be defined.



Good for you.

So, why is division by zero undefined?

The concept, division by zero, is so very important to remember because:

- it extends to other concepts such as rational expressions and equations, and
- it also applies to functions, domains, etc.

RATIONAL EXPRESSIONS:

From rational numbers we extend to include variables in the numerator and/or denominator to make rational expressions.

Thus, the numerator and the denominator would be polynomials.

By definition, a rational expression is the quotient of two polynomials and the polynomial in the denominator can not equal zero.

Examples of rational expressions include:

$$\frac{3}{4}, \quad \frac{1}{x}, \quad \frac{3x}{2x-5}, \quad \frac{x^3-8}{x^2+1}$$



Can you determine why $\frac{\sqrt{x}}{x+3}$ is not a rational expression?

ANSWER: The square root of x is not a polynomial.

So, variables in the denominator of a rational expression can have values that make that denominator equal to zero.

The rational expression is undefined when the **denominator equals to zero**.

Let's look at what values make these rational expressions undefined.



$$\frac{3}{4}, \quad \frac{1}{x}, \quad \frac{3x}{2x-5}, \quad \frac{x^3-8}{x^2+1}$$

You can determine the value that makes a rational expression undefined when you set the denominator = 0 and solve.

$\frac{3}{4}$	The denominator is a constant. Since, $4 \neq 0$ no value can make this	Domain: 4
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	rational expression undefined.	
$\frac{1}{x}$	The denominator has a variable. Set it equal to zero. $x = 0$. Thus, 0 makes the rational expression undefined.	Domain: All real numbers except 0.
$\frac{3x}{2x - 5}$	The denominator has a variable. Set it equal to zero and solve for x . $2x - 5 = 0$ $x = (5/2)$ Thus, $(5/2)$ makes the rational expression undefined.	Domain: All real numbers except $(5/2)$.
$\frac{x^3 - 8}{x^2 + 1}$	The denominator has a variable. Set it equal to zero and solve for x . $x^2 + 1 = 0$ $x = \pm i$ Thus, no value, that is real of course, makes the rational expression undefined.	Domain: All real numbers.

SIMPLIFYING RATIONAL EXPRESSIONS:

Simplifying (reducing) rational expressions is the same process as simplifying (reducing) rational numbers.

$$\frac{a}{b} = \frac{c \cdot k}{d \cdot k} \quad \text{where } c \text{ and } k \text{ are factors of } a \text{ and } d \text{ and } k \text{ are factors of } b$$

$$\frac{a}{b} = \frac{c \cdot \cancel{k}}{d \cdot \cancel{k}} \quad \text{divide the expression by the greatest common factor } k$$

$$\frac{a}{b} = \frac{c}{d} \quad \frac{a}{b} \text{ is written in lowest terms as } \frac{c}{d}$$

We say the two fractions are equivalent fractions.

EXAMPLE 1: Write the rational number in simplified form (lowest terms) .

$$\frac{12}{15} = \frac{\cancel{3} \cdot 4}{\cancel{3} \cdot 5} = \frac{4}{5}$$

Simplify

The rule is that both the numerator and denominator are divided by the greatest common factor which is 3.

FACTOR is the key word to avoid any misconceptions. Being able to factor polynomials is a prerequisite to simplify or operate on rational expressions.

EXAMPLE 2: Write the rational expression in simplified form.

$$\begin{aligned} \frac{2x^3 + 10x^2 + 12x}{4x^2 + 12x} &= \frac{2x(x+3)(x+2)}{4x(x+3)} \\ &= \frac{\cancel{(2x)}(\cancel{x+3})(x+2)}{2(\cancel{2x})(\cancel{x+3})} \\ &= \frac{(x+2)}{2} \end{aligned}$$

First, factor the numerator and denominator completely. Notice, that $4x$ is also factored as $2(2x)$.

Divide the common factors, $(2x)$ and $(x+3)$.

Remember that the domain of this expression is the set of all real numbers except 0 and -3.

Use the Zero-Factor Property on the denominator, $4x(x+3) = 0$, to solve for x .

MULTIPLICATION and DIVISION OF RATIONAL EXPRESSIONS:

The properties to multiply and divide rational numbers should apply to rational expressions.

To multiply rational numbers apply the

properties, $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ where $b \neq 0$, $d \neq 0$.

Verbally, the property states that multiplication of two or more rational numbers is the product of their numerators divided by the product of their denominators.

Remember, always simplify all rational numbers or expressions.

To divide rational numbers apply the

property, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$
 $= \frac{ad}{bc}$ where $b \neq 0$, $c \neq 0$, $d \neq 0$.

Verbally, the property states that the fraction used to divide by is to be inverted (reciprocal) and then used to multiply.

EXAMPLE 1: Multiply and write product in simplified form.

$$\begin{aligned} 24xy^5 \cdot \frac{4x^3}{3y^2} &= \frac{24xy^5}{1} \cdot \frac{4x^3}{3y^2} \\ &= \frac{3 \cdot 8 \cdot 4x^{1+3}y^5}{3y^2} \\ &= \frac{\cancel{3} \cdot 8 \cdot 4x^4y^{5-2}}{\cancel{3}y^{2-2}} \\ &= 32x^4y^3 \end{aligned}$$

Notice the $24xy^5$ is a rational expression whose denominator is understood 1.

Factor the integers use either prime numbers or common factors.

Multiply x's
using the
Product Rule-
Add the
exponents.
Divide y's
using the
Quotient
Rule -
Subtract
exponents.

Divide the
common
factors, 3 and
 y^2 .

Remember,
 $y \neq 0$.

Rewrite the
division
problem as a
multiplication
problem
using the
reciprocal of
the second
rational
expression.

EXAMPLE 2: Divide and write quotient in simplified form.

Notice that
the variables
of the two
rational
expression
are not
written in the
same order.
Use
commutative

$$\begin{aligned}
\frac{b-a}{b+a} \div \frac{a^2-2ab+b^2}{a^2+2ab+b^2} &= \frac{b-a}{b+a} \cdot \frac{a^2+2ab+b^2}{a^2-2ab+b^2} \\
&= \frac{-a+b}{a+b} \cdot \frac{(a+b)(a+b)}{(a-b)(a-b)} \\
&= \frac{-1(a-b)}{a+b} \cdot \frac{(a+b)(a+b)}{(a-b)(a-b)} \\
&= \frac{-1(a+b)}{(a-b)} \\
&= -\frac{a+b}{a-b}
\end{aligned}$$

property to rearrange the variables.

Factor all of the expressions completely. Factor a (-1) from polynomials with a leading negative.

Divide all of the common factors, (a - b), (a + b).

Remember, $a \neq b$ or $a \neq -b$

ADDITION AND SUBTRACTION OF RATIONAL EXPRESSIONS:

For addition and subtraction, the things to be counted must be the same.

$ \begin{array}{r} 2 \text{ apples} \\ + 5 \text{ apples} \\ \hline 7 \text{ apples} \end{array} $	$ \begin{array}{r} \frac{2}{7} = 2 \text{ sevenths} \\ + \frac{4}{7} = 4 \text{ sevenths} \\ \hline \frac{6}{7} = 6 \text{ sevenths} \end{array} $
$ \begin{array}{r} 2x = 2 \text{ x's} \\ - 5x = + (-5) \text{ x's} \\ - 3x = (-3) \text{ x's} \end{array} $ <p>Notice how the subtraction problem is rewritten as an addition problem. Instead of subtracting, you add the inverse.</p>	$ \begin{array}{r} -12\sqrt{3} = \text{negative 12 squareroots of 3} \\ + 5\sqrt{3} = \text{plus 5 squareroots of 3} \\ \hline -7\sqrt{3} = \text{negative 7 squareroots of 3} \end{array} $

So, what if things are not the same. Then, if possible, the things can be renamed or categorized differently to count them.

You have:

5 apples + 3 pears

we rename it to

5 pieces of fruit + 3 pieces of fruit

In mathematics, we say that we find equivalent fractions. The process is like writing rational numbers in simplified form but in reverse order.

When finding equivalent fractions, you always know the denominators. So, it makes it easier to find the common factor.

Once you have found the common factor, multiply both the numerator and denominator by this common factor to get the equivalent fraction.

$$\frac{a}{b} = \frac{?}{d} \Rightarrow b \cdot k = d \quad \text{We always know the 2 denominators. So, find a factor } k \text{ such that } b \text{ times } k = d.$$
$$\frac{a \cdot k}{b \cdot k} = \frac{?}{d} \Rightarrow a \cdot k = c \quad \text{Now, instead of dividing to simplify, we multiply both numerator and denominator by the same factor } k.$$
$$\frac{a \cdot k}{b \cdot k} = \frac{c}{d} \quad \text{The product } \frac{c}{d} \text{ is an equivalent fraction of } \frac{a}{b}.$$

The denominator of the incomplete fraction is called a least common denominator (LCD).

It is found by multiplying the prime factors of two or more denominators.

EXAMPLE 1:

$$\frac{8}{15} + \frac{5}{18} =$$

The two denominators, 15 and 18 are not the same, so a common denominator (LCD) has to be found.

Prime factors of 15 are $3 \cdot 5$, and prime factors of 18 are $2 \cdot 3 \cdot 3$.

Thus, the least common denominator (LCD) is the product of $3 \cdot 5 \cdot 2 \cdot 3 = 90$

Notice, that factors that repeat (pair up) are represented only once.

Only one set of 3's paired up.

The equivalent fractions can be found as follows:

$$\frac{8}{15} + \frac{5}{18} = \frac{?}{90}$$

VERTICAL ILLUSTRATION:

$$\frac{8 \cdot 6}{15 \cdot 6} = \frac{?}{90}$$

We know that the least common denominator is 90.

$$\frac{8 \cdot 6}{15 \cdot 6} = \frac{48}{90}$$

So, the common factor must be 6 because $15 \cdot 6 = 90$.
Now multiply both numerator and denominator by 6.

$$\frac{8}{15} = \frac{48}{90}$$

The equivalent fractions are $\frac{8}{15}$ and $\frac{48}{90}$

$$\frac{5}{18} = \frac{?}{90}$$

Again, the common denominator is known to be 90.

$$\frac{5 \cdot 5}{18 \cdot 5} = \frac{?}{90}$$

So, this common factor must be 5 because $18 \cdot 5 = 90$.

$$\frac{5 \cdot 5}{18 \cdot 5} = \frac{25}{90}$$

Now multiply both numerator and denominator by 5.

$$\frac{5}{18} = \frac{25}{90}$$

The equivalent fractions are $\frac{5}{18}$ and $\frac{25}{90}$

HORIZONTAL ILLUSTRATION:

$$\frac{8}{15} + \frac{5}{18} = \frac{?}{90}$$

Both fractions are processed at the same time in a step-by-step process.

$$\frac{8 \cdot 6}{15 \cdot 6} + \frac{5 \cdot 5}{18 \cdot 5} = \frac{?}{90}$$

Each fraction is multiplied by its common factor.

$$\frac{48}{90} + \frac{25}{90} = \frac{?}{90}$$

Simplifying the multiplication process.

$$\frac{48}{90} + \frac{25}{90} = \frac{73}{90}$$

Notice that the sum of the two fractions can now be easily determined by adding the numerators and keeping the same denominator.

Let's try subtraction of rational expressions because the problem can be rewritten as an addition problem when you add the inverse of the subtrahend.

PROCESS

EXAMPLE 1:

$$\begin{aligned} & \frac{x+2}{x^2-x-2} - \frac{x-1}{x^2-2x-3} \\ &= \frac{x+2}{x^2-x-2} + \frac{-(x-1)}{x^2-2x-3} \\ &= \frac{x+2}{(x-2)(x+1)} + \frac{-(x-1)}{(x-3)(x+1)} \\ &= \frac{(x+2)(x-3)}{(x-2)(x+1)(x-3)} + \frac{-(x-1)(x-2)}{(x-3)(x+1)(x-2)} \\ &= \frac{x^2-x-6}{(x-2)(x+1)(x-3)} + \frac{-(x^2-3x+2)}{(x-3)(x+1)(x-2)} \\ &= \frac{x^2-x-6}{(x-2)(x+1)(x-3)} + \frac{-x^2+3x-2}{(x-3)(x+1)(x-2)} \\ &= \frac{x^2-x-6-x^2+3x-2}{(x-2)(x+1)(x-3)} \\ &= \frac{2x-8}{(x-2)(x+1)(x-3)} \\ &= \frac{2(x-4)}{(x-2)(x+1)(x-3)} \end{aligned}$$

1. Rewrite the problem as addition of the inverse of the subtrahend. $-(x-1)$ is the additive inverse of $(x-1)$

2. Completely factor the denominators. LCD = $(x-2)(x+1)(x-3)$

3. Multiply both numerator and denominator of each term by its common factor. $(x-3)$ is the common factor of the first term. $(x-2)$ is the common factor of the second term.

4. Multiply the polynomials in the numerators. Be careful with the negative.

5. Now, distribute the negative.

6. Combine like terms in the numerators and keep the same denominator.

7. Simplify.

8. Factor the numerator

9. Rational expression is in simplified form.

Let's try one more example. We have a function that is the sum of two rational expressions.

$$\begin{aligned}
 f(x) &= \frac{1}{x} + \frac{x-2}{x-6} \\
 &= \frac{1(x-6)}{x(x-6)} + \frac{x(x-2)}{x(x-6)} \\
 &= \frac{x-6}{x(x-6)} + \frac{x^2-2x}{x(x-6)} \\
 &= \frac{x^2-x-6}{x(x-6)} \\
 &= \frac{(x-3)(x+2)}{x(x-6)} \quad \text{or} \quad \frac{x^2-x-6}{x^2-6x}
 \end{aligned}$$

Add rational expressions.

$$\text{LCD} = x(x-6)$$

Multiply the factors.

Combine like terms.

Factor numerator to simplify.
Or, if you prefer, multiply factors in the denominator.

The numerator and denominator of the rational expressions can be written in factored form or as polynomials with real coefficients.

COMPLEX FRACTIONS:

A complex fraction is a fraction in which the numerator or the denominator, or both, involves a fraction.

$$\frac{\frac{a}{b}}{\frac{c}{d}}$$

Is a complex fraction. Notice, both the numerator and the denominator are fractions.

Two methods to simplify a complex fraction will be illustrated.

Method #1:

Remember, a fraction (rational expression) is the quotient of two polynomials. Thus, the method involves rewriting the fraction to show division using the symbol for division, \div .

But, first the numerator and denominator must be simplified to obtain a single fraction in each.

$$\frac{\frac{a}{b}}{\frac{c}{d}}$$

In general, $\frac{\frac{a}{b}}{\frac{c}{d}}$ can be written as $(\frac{a}{b}) \div (\frac{c}{d})$

EXAMPLE 1:

$$\frac{\frac{2}{3} + \frac{1}{y}}{\frac{4}{9} - \frac{1}{y^2}} = \frac{\frac{2y}{3y} + \frac{1(3)}{3y}}{\frac{4y^2}{9y^2} + \frac{-1(9)}{9y^2}}$$

Notice, the additive inverse

Simplify:
Numerator and
Denominator

LCD of numerator = $3y$
LCD of denominator = $9y^2$

$$= \frac{\frac{2y+3}{3y}}{\frac{4y^2-9}{9y^2}}$$

Multiply and combine like terms.

$$= \frac{2y+3}{3y} \div \frac{4y^2-9}{9y^2}$$

Rewrite fraction using the symbol for division.

$$= \frac{2y+3}{3y} \cdot \frac{9y^2}{4y^2-9}$$

Multiply by the reciprocal.

$$= \frac{\cancel{2y+3}}{\cancel{3y}} \cdot \frac{\cancel{3y(3y)}}{(2y-3)\cancel{(2y+3)}}$$

Factor completely and divide the common factors, $3y$ and $(2y+3)$

$$= \frac{3y}{2y-3}$$

METHOD # 2:

In this method, each term of the complex fraction is multiplied by the least common denominator of all the fractions in the numerator and denominator.

$$\frac{\frac{2}{3} + \frac{1}{y}}{\frac{4}{9} - \frac{1}{y^2}} = \frac{\left(\frac{9y^2}{1}\right)\frac{2}{3} + \left(\frac{9y^2}{1}\right)\frac{1}{y}}{\left(\frac{9y^2}{1}\right)\frac{4}{9} + \left(\frac{9y^2}{1}\right)\frac{-1}{y^2}}$$

$$= \frac{6y^2 + 9y}{4y^2 - 9}$$

$$= \frac{3y(2y+3)}{(2y-3)(2y+3)}$$

$$= \frac{3y}{2y-3}$$

LCD of all fractions
 $3, (y), 9, (y^2) = 9y^2$

Multiply each term by $9y^2$

Multiply and combine like terms.

Factor completely and divide the common factor $(2y+3)$