

3.2. SOLVING RATIONAL EQUATIONS

What is the difference between a **rational expression** and a **rational equation**?

A rational expression is an incomplete mathematical sentence.

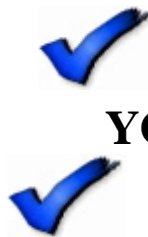
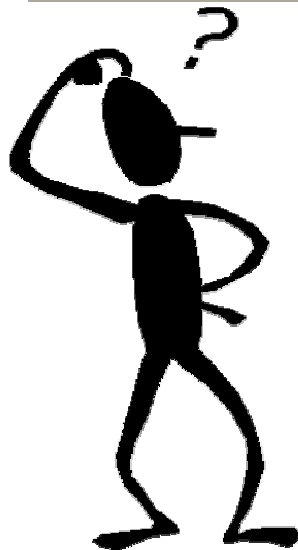
Rational expressions are simplified.

A rational equation is a complete mathematical (algebraic) sentence.

If a rational equation contains a variable, the equation is solved for that variable.

EXAMPLE:

EXPRESSION	EQUATION
$\frac{3}{5} + \frac{2}{3}$	$\frac{7}{8} + \frac{1}{5} = n$
$\frac{4x-1}{2x-6} - \frac{2x+4}{x}$	$\frac{3x^2 - x + 5}{2x+6} = \frac{x^2 - 4}{x^2 - 9}$



**EXPRESSIONS,
YOU SIMPLIFY.**

**EQUATIONS, YOU
SOLVE**

SOLVING A RATIONAL EQUATION

To solve a rational equation involves one main concept. The rest of the process is solving basic equations. Remember, one of the methods to simplify complex fractions involved multiplication by the Least Common Denominator (LCD) of all fractions.



What happens when you multiply each fraction by the LCD?

LOOK CLOSER AT THAT METHOD.



$$\frac{\frac{2}{3} + \frac{1}{y}}{\frac{4}{9} - \frac{1}{y^2}} = \frac{\left(\frac{9y^2}{1}\right)\frac{2}{3} + \left(\frac{9y^2}{1}\right)\frac{1}{y}}{\left(\frac{9y^2}{1}\right)\frac{4}{9} + \left(\frac{9y^2}{1}\right)\frac{-1}{y^2}}$$

$$= \frac{6y^2 + 9y}{4y^2 - 9}$$

$$= \frac{3y(2y+3)}{(2y-3)(2y+3)}$$

$$= \frac{3y}{2y-3}$$

LCD of all fractions
 $3, (y), 9, (y^2) = 9y^2$

Multiply each term by $9y^2$

Multiply and combine like terms.

Factor completely and divide the common factor $(2y+3)$

CAN YOU EXPLAIN WHAT HAPPENED?

THAT IS CORRECT....

All denominators from the numerator and denominator were eliminated leaving a single fraction. Thus, a rational expression.

Well, we will apply the same idea to solving rational equations.

Multiply **each term** of the equation and that includes all terms to the left and to the right of the equal sign.

EXAMPLE : SOLVE THE RATIONAL EQUATION.

PROBLEM

$$\frac{4}{2x} - 1 = \frac{5}{x}$$

Rational Equation

$$\frac{4}{2x} - \frac{1}{1} = \frac{5}{x}$$

Express the 1 as a rational number.

$$\left(\frac{2x}{1}\right)\frac{4}{2x} - \left(\frac{2x}{1}\right)\frac{1}{1} = \left(\frac{2x}{1}\right)\frac{5}{x}$$

Denominators: $2x, 1, x$

LCD: $2x$

Multiply each term by the LCD.

$$4 - 2x = 10$$

Notice, all denominators have been eliminated.

$$-2x = 10 - 4$$

Additive inverse of 4.

$$\frac{-2x}{-2} = \frac{6}{-2}$$

Multiplicative inverse of -2

$$x = -3$$

Solution Set. $x = \{-3\}$

CHECK

ALWAYS CHECK YOUR ANSWER

$$\frac{4}{2x} - 1 = \frac{5}{x}$$

Always, use the original equation

$$\frac{4}{2(-3)} - \frac{1}{1} = \frac{5}{(-3)}$$

Substitute the answer.

$$-\frac{2}{3} - \frac{3}{3} = -\frac{5}{3}$$

You can use your calculator to verify that the two expressions equal.

$$-\frac{5}{3} = -\frac{5}{3} \checkmark$$

VERIFIED: Answer is correct.

EXAMPLE 2: SOLVING PROBLEMS THAT HAVE EXTRANEOUS ANSWERS

P R O B L E M

$$\frac{x-2}{x+4} + \frac{x+1}{x+6} = \frac{11x+32}{x^2+10x+24}$$

$$\frac{x-2}{x+4} + \frac{x+1}{x+6} = \frac{11x+32}{(x+4)(x+6)}$$

$$\left(\frac{(x+4)(x+6)}{1}\right)\frac{x-2}{x+4} + \left(\frac{(x+4)(x+6)}{1}\right)\frac{x+1}{x+6} = \left(\frac{(x+4)(x+6)}{1}\right)\frac{11x+32}{(x+4)(x+6)}$$

$$(x+6)(x-2) + (x+4)(x+1) = 11x+32$$

$$x^2 + 4x - 12 + x^2 + 5x + 4 = 11x + 32$$

$$2x^2 + 9x - 8 = 11x + 32$$

$$2x^2 + 9x - 11x - 8 - 32 = 0$$

$$2x^2 - 2x - 40 = 0$$

$$2(x^2 - x - 20) = 0$$

$$\frac{2(x^2 - x - 20)}{2} = \frac{0}{2}$$

$$x^2 - x - 20 = 0$$

$$(x-5)(x+4) = 0$$

$$x-5 = 0 \Rightarrow x = 5$$

$$x+4 = 0 \Rightarrow x = -4$$

P R O C E S S

Completely factor the denominators.

Multiply each term by the LCD, $(x+4)(x+6)$

Divide common factors.

Denominators eliminated.

Multiply binomials.

Simplify by combining like terms.

Additive inverse of $11x$ and 32 to make quadratic equal 0.

Simplify by combining like terms.

2 is a common factor of the trinomial.

Multiplicative inverse of 2.

Factor the trinomial.

Apply the Zero-Factor Property

Solutions are $x = 5$ and $x = -4$.

Check your answers.

CHECKING ANSWERS

CHECK: Let $x = 5$

$$\frac{x-2}{x+4} + \frac{x+1}{x+6} = \frac{11x+32}{x^2+10x+24} \quad \text{Use original equation.}$$

$$\frac{5-2}{5+4} + \frac{5+1}{5+6} = \frac{11(5)+32}{(5)^2+10(5)+24} \quad \text{Substitute 5 for every } x$$

$$\frac{3}{9} + \frac{6}{11} = \frac{87}{99} \quad \text{Simplify}$$

$$\frac{33}{99} + \frac{54}{99} = \frac{87}{99} \quad \text{The equation is verified.}$$

$$\frac{87}{99} = \frac{87}{99} \quad \checkmark \quad \text{The answer is correct.}$$

CHECK: Let $x = (-4)$

$$\frac{x-2}{x+4} + \frac{x+1}{x+6} = \frac{11x+32}{x^2+10x+24} \quad \text{Again, use the original equation.}$$

$$\frac{(-4)-2}{(-4)+4} + \frac{(-4)+1}{(-4)+6} = \frac{11(-4)+32}{(-4)^2+10(-4)+24} \quad \text{Substitute } (-4) \text{ for every } x$$

$$\frac{-6}{0} + \frac{-3}{2} = \frac{-12}{0}$$

Ooops!!!
Division by zero

The answer (-4) is an extraneous value. It is not in the domain of the equation.

SO, THE SOLUTION SET OF THIS EQUATION IS: $x = \{5\}$

THE DOMAIN OF THIS EQUATION IS THE SET OF ALL REAL NUMBERS EXCEPT -4 AND -6 .

EXAMPLE 3:

When a rational equation is made up of 2 fractions such as

$$\frac{a}{b} = \frac{c}{d},$$

we have what we call a proportion.

To eliminate denominators, multiply by the LCD, bd , .

$$\left(\frac{\cancel{bd}}{1}\right)\frac{a}{\cancel{b}} = \left(\frac{\cancel{bd}}{1}\right)\frac{c}{\cancel{d}}$$

The result is an equation $ad = bc$.

This process is called solving rational equations by using **CROSS-PRODUCTS**.

The process of cross multiplication can be applied in the following example.

PROBLEM

$$\frac{2}{v-1} = \frac{5}{v+3}$$

TWO EQUAL FRACTIONS

$$2(v+3) = 5(v-1) \quad \text{Apply the cross-products rule}$$

$$2v + 6 = 5v - 5 \quad \text{Distribute}$$

$$2v - 5v = -5 - 6 \quad \text{Additive inverse of } 5v \text{ and } 6$$

$$-3v = -11 \quad \text{Simplify}$$

$$\frac{-3v}{-3} = \frac{-11}{-3} \quad \text{Multiplicative inverse of } -3$$

$$v = \frac{11}{3} \quad \text{Solution.}$$

PROBLEM CHECK

$$\frac{2}{v-1} = \frac{5}{v+3} \quad \text{Check using original equation}$$

$$\frac{2}{\frac{11}{3} - \frac{3}{3}} = \frac{5}{\frac{11}{3} + \frac{3}{3}} \quad \begin{array}{l} \text{Substitute } (\frac{11}{3}) \text{ for every } x \\ \text{Notice you have a complex fraction.} \end{array}$$

$$\frac{2}{\frac{8}{3}} = \frac{5}{\frac{20}{3}} \quad \text{Simplify}$$

$$2\left(\frac{3}{8}\right) = 5\left(\frac{3}{20}\right) \quad \text{Invert and multiply}$$

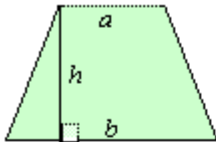
$$\frac{3}{4} = \frac{3}{4} \quad \checkmark \quad \text{Checks. The answer is correct.}$$

LITERAL EQUATIONS

Solving rational equations also involves solving literal equations which are formula-type equations containing multiple variables. These equations can be manipulated to solve for a specific variable.

The idea is to follow the order of operations in reversed order.

Formula for area of trapezoid
Solve for b



$$A = \frac{1}{2}(a + b)h; \quad \text{for } b$$

Area is equal to one half the sum of the two bases times the height.

$$2A = \cancel{(2)} \frac{1}{\cancel{2}}(a + b)h$$

To eliminate the $(\frac{1}{2})$ multiply by 2

$$\frac{2A}{h} = \frac{(a + b)\cancel{h}}{\cancel{h}}$$

Multiplicative inverse of h

$$\frac{2A}{h} = a + b$$

Remove parenthesis all factors have been eliminated

$$a + b = \frac{2A}{h}$$

Property of symmetry to get variable b on the left side of the equal sign

$$b = \frac{2A}{h} - a$$

Additive inverse of a
This is the answer

$$b = \frac{2A - ah}{h}$$

Or simplify expression and

Write the equation as a single fraction

APPLICATIONS INVOLVING RATIONAL EXPRESSIONS

You have learned from previous mathematics courses to solve worded problems.

The examples presented here are those that involve rational equations.

Examine this worded-problem strategy.

STEP 1: Because Mathematics is like another language (like English or Spanish), translating English to Mathematics is similar to translating it to any other language. English sentences that contain mathematics translate into mathematical equations.

So read one sentence and translate into an algebraic equation. The verb of an English sentence is usually the equal sign of the algebraic equation.

STEP 2: Select a variable to represent the unknown quantity.

STEP 3: Use drawings, tables, diagrams, etc. to help you understand the context of the problem. You may need to refresh your memory if a specific formula applies. For instance,

1. if something is moving (airplane, train, boats, cyclists, etc.), use the distance formula: $D = RT$
2. if geometric (rectangles, triangles, etc.), use perimeter or area formulas
3. if money (investments), use $I = PRT$
4. if mix (chemicals, candies, etc.), use percentage (cost) = rate (quantity)
5. if work related, use work = rate (time)

STEP 4: Solve the algebraic equation.

STEP 5: Very important to check the answer. Substitute the answer into the original equation. Eliminate any extraneous answers.

EXAMPLE 1:

MOTION PROBLEM



Step 1: **READ** the first sentence. **ONLY THE FIRST SENTENCE.**

Two boats that have the same speed in still water travel in opposite directions on a river with a current of 8 kilometers per hour (kph). After a certain period of time one boat is 24 kilometers upstream and the other is 56 kilometers downstream. What is the speed of the boat in still water?

Let's put ourselves in the context of the problem:

Draw a sketch or picture of the boats in a river. The following questions lead from the **first sentence**.

Analysis

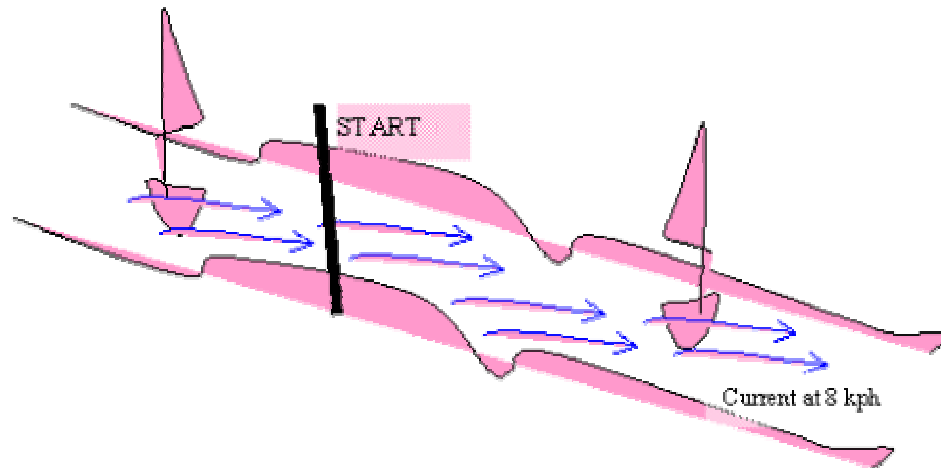
1. Boats have same speed: **What speed?**

So, we select **R** (for rate) to represent the speed of the boat in still water.

Let **R** = speed of boat in still water.

2. Current has a speed of 8 kph.
3. Prerequisite knowledge: Distance formula.

$$\text{Distance} = \text{Rate times Time} \quad D = R \cdot T$$



Step 2: Now, continue to the next page and read only the second sentence.



Step 2: R E A D the second sentence. Continue to analyze.

Two boats that have the same speed in still water travel in opposite directions on a river with a current of 8 kilometers per hour (kph). After a certain period of time one boat is 24 kilometers upstream and the other is 56 kilometers downstream. What is the speed of the boat in still water?

Analysis

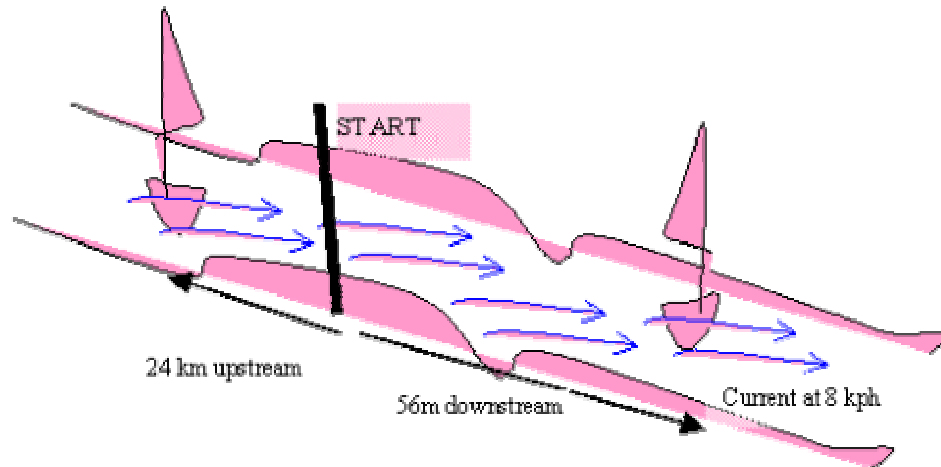
1. 24 km. upstream. This boat goes slower.
How much slower?
 The current is slowing the boat, so the boat goes 8 km. slower.
2. 56 km. downstream. This boat goes faster.
How much faster?
 The current is speeding the boat, so the boat goes 8 km. faster.
3. We say 8 km. slower or faster but, **slower or faster than what?**
 Slower or faster than the boat traveling in still water.
4. Represent as an algebraic expression.
 upstream (boat goes slower): $R - 8$
 downstream (boat goes faster): $R + 8$
5. After a certain period of time **implies that the two boats traveled the same**

amount of time.

Thus, you have the equation

$$\text{Time}_{\text{slower boat}} = \text{Time}_{\text{faster boat}}$$

The rest is substituting known values into the equation and solving for the unknown quantity.



EQUATION: $\text{Time}_{\text{slower boat}} = \text{Time}_{\text{faster boat}}$

MANIPULATE THE LITERAL EQUATION:

6. Solve the distance formula (literal equation) for Time.

$$D = RT, \text{ for } T$$

$$\frac{D}{R} = \frac{\cancel{RT}}{\cancel{R}}$$

$$T = \frac{D}{R}$$

This Time applies to each boat

7. SUBSTITUTE the known values into the equation.

$$\text{Time}_{\text{slower boat}} = \text{Time}_{\text{faster boat}}$$

$$\frac{D_s}{R_s} = \frac{D_f}{R_f} \quad \text{Notice the subscripts - they represent the distance and rate of each boat.}$$

$$\frac{24}{R-8} = \frac{56}{R+8} \quad \begin{array}{l} \text{slower boat (} D_s \text{ and } R_s) \\ \text{faster boat (} D_f \text{ and } R_f) \end{array}$$

8. SOLVE the equation.

$$24(R+8) = 56(R-8) \quad \text{cross products}$$

$$24R + 192 = 56R - 448 \quad \text{distributive property}$$

$$24R - 56R = -448 - 192 \quad \text{additive inverse } 56R \text{ and } 192$$

$$\frac{-32R}{-32} = \frac{-640}{-32} \quad \text{simplify operations}$$

$$R = 20 \quad \text{multiplicative inverse of } -32$$

R = 20 speed of boat in still water.



Step 3: **READ** the third sentence. This is your question sentence and we need to determine if the solution to the variable answers the question.

Two boats that have the same speed in still water travel in opposite directions on a river with a current of 8 kilometers per hour (kph). After a certain period of time one boat is 24 kilometers upstream and the other is 56 kilometers downstream. **What is the speed of the boat in still water?**

9. **ANSWER:** R = 20 kph in still water.

The question is to find the speed of the boat in still water.

They could have asked:

What is the speed of the boat going upstream?

$$R - 8 = 20 - 8 = 12 \text{ kph}$$

What is the speed of the boat going downstream?

$$R + 8 = 20 + 8 = 28 \text{ kph}$$

So, we must **READ** the question carefully in order to answer correctly.

10: **CHECK:**

$$24 / (20 - 8) = 56 / (20 + 8)$$

$$24 / 12 = 56 / 28$$

$$2 =$$

2

substitute into original
equation.

simplify

checks.

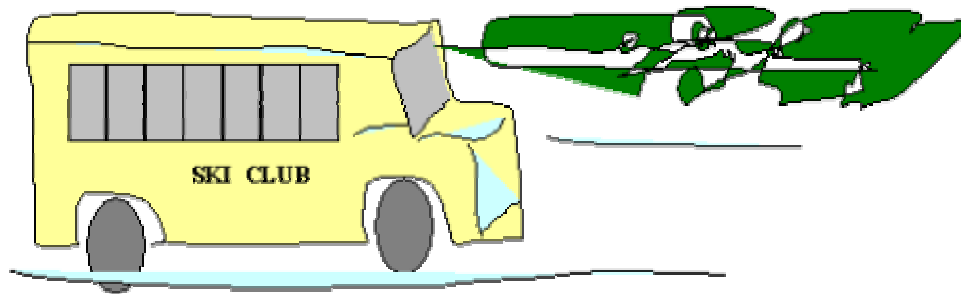
EXAMPLE 2

SHARING THE COST: (Larson & Hostetler, 1997)



Step 1: **READ** the first sentence. **ONLY THE FIRST SENTENCE.**

A ski club chartered a bus for a ski trip at a cost of \$480. In an attempt to lower the bus fare per skier, the club invited nonmembers to go along. After five nonmembers joined the trip, the fare per skier decreased by \$4.80. How many club members are going on the trip?



Total cost at \$480.

If only one person goes, he/she will have to pay \$480.

If two people go, each will pay $(\frac{480}{2}) = \$240$

Since we do not know how many members are going we can let that quantity be represented as x .

Let x = number of members going on the ski trip

Thus, each member pays $(\frac{480}{x})$

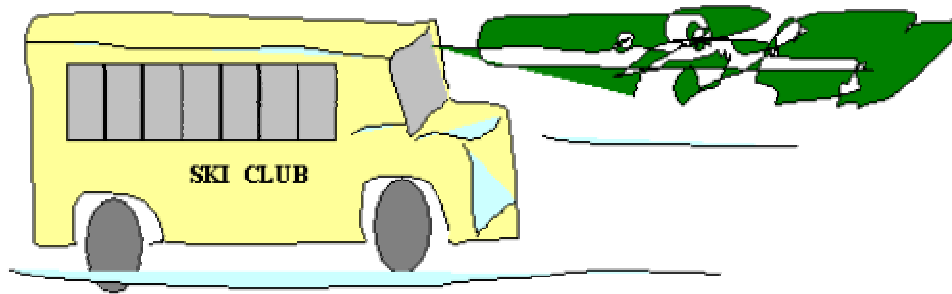


Step 2:

SENTENCE.

READ the second sentence. **ONLY THE SECOND**

A ski club chartered a bus for a ski trip at a cost of \$480. **In an attempt to lower the bus fare per skier, the club invited nonmembers to go along.** After five nonmembers joined the trip, the fare per skier decreased by \$4.80. How many club members are going on the trip?



Total cost at \$480.

If nonmembers go, then the number of people sharing the cost increases. We do not know how many more people are going so we can let another variable represent this number.

Let y = number of nonmembers going.

So, how many people are going? $\text{Members} + \text{Nonmembers} = x + y$

And now the cost per person is $(\frac{480}{x+y})$.



Step 3: **READ** the third sentence. **ONLY THE THIRD SENTENCE.** A ski club chartered a bus for a ski trip at a cost of \$480. In an attempt to lower the bus fare per skier, the club invited nonmembers to go along. **After five nonmembers joined the trip, the fare per skier decreased by \$4.80.** How many club members are going on the trip?

PROBLEM

$$\frac{480}{x} - \frac{480}{x+5} = 4.80$$

$$\left(\frac{x(x+5)}{1}\right)\frac{480}{\cancel{x}} - \left(\frac{x(x+5)}{1}\right)\frac{480}{\cancel{x+5}} = \left(\frac{x(x+5)}{1}\right)4.80$$

$$480(x+5) - 480(x) = 4.8(x)(x+5)$$

$$480x + 2400 - 480x = 4.8x^2 + 24x$$

$$2400 = 4.8x^2 + 24x$$

$$0 = 4.8x^2 + 24x - 2400$$

$$0 = 4.8(x^2 + 5x - 500)$$

$$\frac{0}{4.8} = \frac{\cancel{4.8}(x^2 + 5x - 500)}{\cancel{4.8}}$$

$$0 = x^2 + 5x - 500$$

$$0 = (x + 25)(x - 20)$$

$$x + 25 = 0 \Rightarrow x = -25$$

$$x - 20 = 0 \Rightarrow x = 20$$

PROCESS

LCD = $x(x+5)$

Multiply each term by LCD

Multiply numerators

Simplify multiplication

Combine like terms

Additive inverse of 2400

4.8 common factor

Multiplicative inverse of 4.8

Simplify

Factor the trinomial
Apply Zero-Factor Property

CHECKING ANSWERS

CHECK: Let $x = -25$

$$\frac{480}{x} - \frac{480}{x+5} = 4.8$$

$$\frac{480}{-25} - \frac{480}{-25+5} = 4.8$$

$$\frac{480}{-25} - \frac{480}{-20} = 4.8$$

$$4.8 = 4.8 \quad \checkmark$$

CHECK: Let $x = 20$

$$\frac{480}{x} - \frac{480}{x+5} = 4.8$$

$$\frac{480}{20} - \frac{480}{20+5} = 4.8$$

$$\frac{480}{20} - \frac{480}{25} = 4.8$$

$$4.8 = 4.8 \quad \checkmark$$

Although both answers check, the question is **how many members are going on the trip?**

Well, we know that -25 people cannot be a reasonable answer. So, this answer must be excluded.

ANSWER: 20 members of the Ski Club are going on the trip.

A ski club chartered a bus for a ski trip at a cost of \$480. In an attempt to lower the bus fare per skier, the club invited nonmembers to go along. After five nonmembers joined the trip, the fare per skier decreased by \$4.80. How many club members are going on the trip?

ANSWER: 20 members of the Ski Club are going on the trip

What if you were asked the following questions?

1. How much would each member have paid if nonmembers were not invited?
2. How many people in all (members and nonmembers) went on the trip?
3. What fare did each person pay to go on the trip?
4. What is the difference of the two fares between members only and total people on the trip?
5. What would the fare have been if 10 nonmembers had joined the trip?

ANSWERS TO THE QUESTIONS ON THE PREVIOUS PAGE.

1. How much would each member have paid if nonmembers were not invited? **Each member would have paid \$24.**
2. How many people in all (members and nonmembers) went on the trip? **A total of 25 people went on the trip.**
3. What fare did each person pay to go on the trip? **Each person paid a fare of \$19.50**
4. What is the difference of the two fares between members only and total people on the trip? **The difference between \$24 and \$19.20 is \$4.80**
5. What would the fare have been if 10 nonmembers had joined the trip? **If 10 nonmembers joined the trip, each person on the trip would pay \$16.00**

EXAMPLE 3:

WORK RELATED PROBLEM

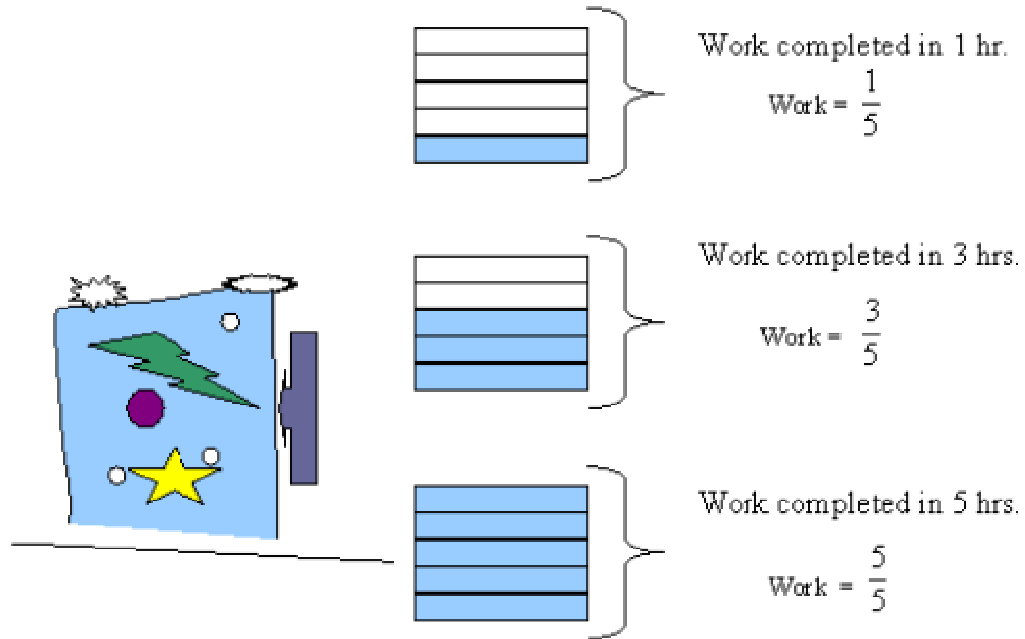
WORK = RATE times **TIME**

That means that the amount of work done is determined by the **multiplying** the rate at which the work is done in 1 unit of time **and** the number of units worked.

For instance, if it takes a machine to complete a job in 5 hours, how much of the work is done in 3 hours?

ONE COMPLETE JOB DONE IN 5 HOURS

$$\text{WORK} = \frac{1}{\text{hours to complete the job}} \left(\text{hours worked} \right)$$



A faster machine can complete the same job in 4 hours, how much of the work is done in 3 hours?

ONE COMPLETED JOB IN 4 HOURS

$$\text{WORK} = \frac{1}{\text{hours to complete the job}} \left(\text{hours worked} \right)$$

