

### 3.3. GRAPHS OF RATIONAL FUNCTIONS

In a previous lesson you learned to sketch graphs by understanding what controls their behavior.

Some of those sketching aids include:

y-intercept (if any)	A y-intercept is found by evaluating $f(0)$ .
x-intercept(s) (if any)	x-intercept(s) are found by setting $f(x) = 0$
negative leading coefficient	If the leading coefficient is negative, the graph of $f(x)$ is reflected on the x-axis.

New sketching aids include:

vertical asymptote (if any)	Set the denominator equal to zero and solve.
horizontal asymptote (if any) or oblique asymptote	Use the rule that applies to the relationship of the degree of the numerator and the degree of the denominator based on the trichotomy property for real numbers.
points in each interval of the domain	Find points in each interval to complete the graph of $f(x)$ .

## USING A GRAPHING CALCULATOR

The use of a graphing calculator can be helpful in sketching rational functions.

- The use of **grouping symbols** (parenthesis) is so very important to let the calculator know where a numerator starts and ends. The same goes for the denominator.
- The graphing calculator does not have strong capabilities to draw a precise graph making it difficult to interpret.
- Because of the vertical and horizontal/oblique asymptotes of rational functions, sections of this graph may appear to be connected.
- Vertical asymptotes are displayed but not the horizontal/oblique asymptotes.
- Using the table of ordered pairs related to a graph can reduce the problem of interpreting graphs.
- Another option is to change the mode of the graphing calculator to dot mode so that the curve is displayed using dots rather than a smooth curve.

EXAMPLE 1: THE GRAPH OF THE SIMPLEST RATIONAL FUNCTION

Sketch the graph of

$$f(x) = \frac{1}{x}$$

IMPLEMENTING  
SPECIFIC  
CHARACTERISTICS

1. Find the y-intercept. Evaluate  $f(0)$ .

$f(0) = 1/0$  the rational function is undefined for  $x = 0$ .

**$f(x)$  has no y-intercept.**

2. Find the x-intercept(s). Set  $f(x) = 0$  and solve the equation.

$$f(x) = \frac{1}{x}$$

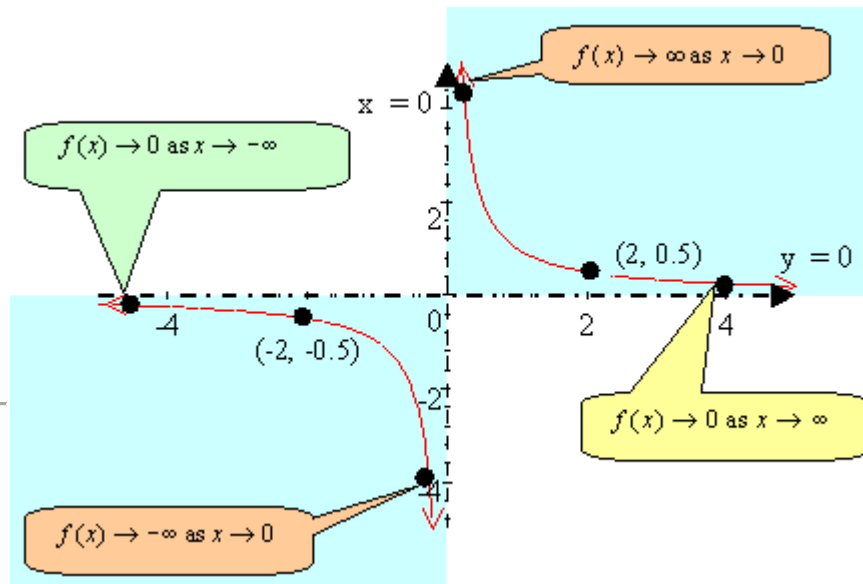
$$0 = \frac{1}{x}$$

$$\left(\frac{x}{1}\right)0 = \frac{1}{\cancel{x}}\left(\frac{x}{1}\right)$$

$$0 \neq 1$$

This equation is a contradiction.

**$f(x)$  has no x-intercepts.**



Notice the points marked by the callouts.

Notice, you multiply both sides of the equation by the denominator.

The left side will always equal zero.

The right side will contain only the numerator since the denominator is eliminated.

So,  $x$ -intercepts are found by setting the numerator = 0 and solving that equation.

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3. The leading coefficient of  $x$  is positive so, the graph of  $f(x)$  will be in the positive Quadrant I and Quadrant III.

When the leading coefficient of  $x$  is negative, the graph of  $(x)$  will be in the negative quadrants, Quadrant II and Quadrant IV.

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Graphing asymptotes:

4. Vertical Asymptote. Let the denominator equal zero and solve.

A sketching aid is that once you have found a point in each interval of the domain, you know that a section of the graph will be in that particular region formed by the two asymptotes. These regions are illustrated by the light blue rectangles shown in the figure above.

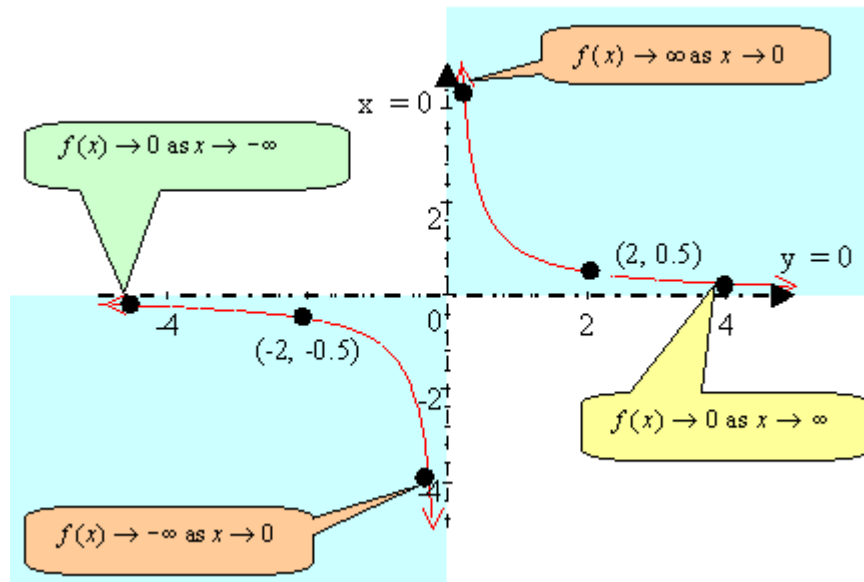
$x = 0$  or the  $y$  axis is the vertical asymptote.

So, place a point next to the asymptotes that form that region. Then draw a smooth curve through the points.

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5. Horizontal Asymptote.

Since the degree of the numerator is less than the degree of the denominator,  $0 < 1$ , the horizontal asymptote is  $y = 0$  or the  $x$  axis.



Notice the points marked by the callouts.

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6. Find points in each interval of the domain. Domain =  $(-\infty, 0) \cup (0, \infty)$

$x$	-2	2
$f(x)$	-0.5	0.5

EXAMPLE 2: Sketch the graph of

$$f(x) = \frac{3x}{x-4}$$

### STEPS

1. Find the y-intercept. Evaluate  $f(0)$

$$f(0) = 0 \quad \mathbf{(0, 0) \text{ is the y-intercept}}$$

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2. Find the x-intercept. Set numerator = 0 and solve

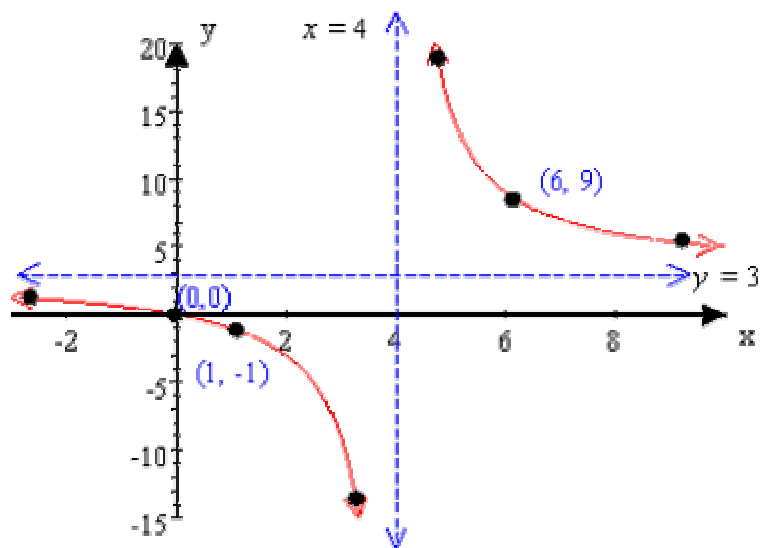
$$\begin{aligned} 3x &= 0 \\ x &= 0 \quad \mathbf{(0, 0) \text{ is the x-intercept}} \end{aligned}$$

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3. Leading coefficient is positive,

**Graph will be in quadrants I and III**

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4. Find the vertical asymptote. Set denominator = 0 and solve

$$x - 4 = 0$$

$$x = 4 \quad \text{Vertical Asymptote}$$

5. Find the horizontal asymptote. Degree of numerator = Degree of denominator

$$y = \frac{3}{1} \text{ leading coefficients of } \frac{P(x)}{Q(x)}$$

$$y = 3 \quad \text{Horizontal Asymptote}$$

6. Find some points in each interval of the domain.

$$\text{Domain} = (-\infty, 4) \cup (4, \infty)$$

$x$	1	6
$f(x)$	-1	9

EXAMPLE 3: Sketch the graph of

$$f(x) = \frac{-x}{x^2 + x - 2}$$

## STEPS

1. Find the y-intercept. Evaluate  $f(0)$

$$f(0) = 0 \quad \mathbf{(0, 0) \text{ is the y-intercept}}$$

2. Find the x-intercept. Set numerator = 0 and solve

$$\begin{aligned} -x &= 0 \\ x &= 0 \quad \mathbf{(0, 0) \text{ is the x-intercept}} \end{aligned}$$

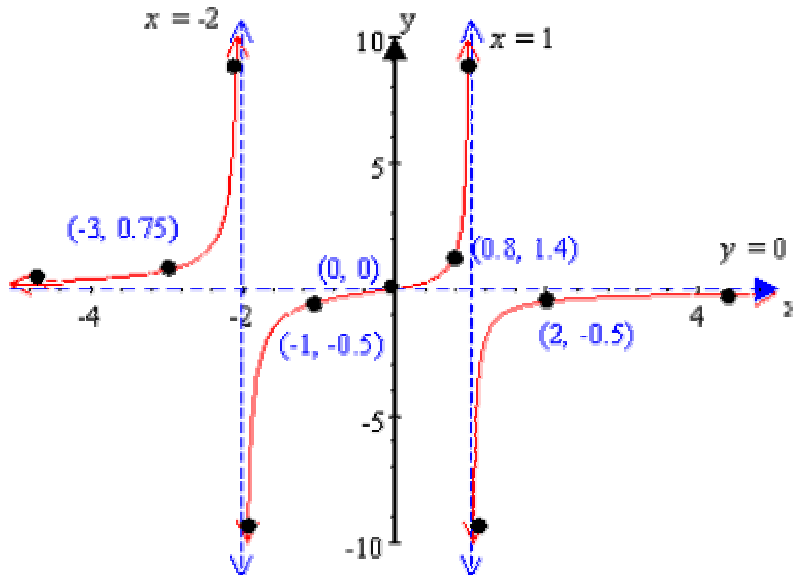
3. Leading coefficient is negative,

**Graph will be in quadrants II and IV.**

4. Find the vertical asymptotes. Set the denominator = 0 and solve.

$$\begin{aligned} x^2 + x - 2 &= 0 \\ (x + 2)(x - 1) &= 0 \\ x + 2 = 0 &\rightarrow x = -2 \quad \mathbf{Vertical Asymptote} \\ x - 1 = 0 &\rightarrow x = 1 \quad \mathbf{Vertical Asymptote} \end{aligned}$$





5. Find the horizontal asymptote. Degree of numerator < Degree of denominator

$y = 0$     **Horizontal Asymptote**

**Notice that  $(0,0)$  is a point of the graph.**

**This graph will cross the horizontal asymptote.**

6. Find some points in each interval of the domain.

$$\text{Domain} = (-\infty, -2) \cup (-2, 1) \cup (1, \infty)$$

Because the graph crosses the horizontal asymptote, we chose to find a point to the left and to the right of  $(0,0)$ .

$x$	-3	-1	0.8	2
$f(x)$	0.75	-0.5	1.4	-0.5

EXAMPLE 4: Sketch the graph of

$$f(x) = \frac{(x-1)(x+3)}{(x-2)}$$

### STEPS

1. Find the y-intercept. Evaluate  $f(0)$

$$f(0) = 1.5 \quad \mathbf{(0, 1.5) \text{ is the y-intercept}}$$

2. Find the x-intercept. Set numerator = 0 and solve

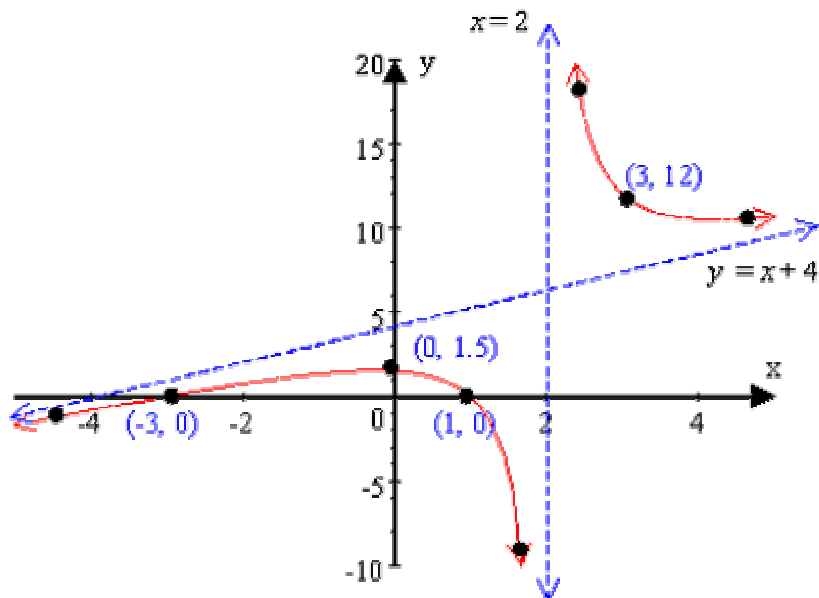
$$(x-1)(x+2) = 0$$
$$x-1 = 0 \rightarrow x = 1 \quad \mathbf{(1, 0) \text{ is an x-intercept}}$$
$$x+2 = 0 \rightarrow x = -2 \quad \mathbf{(-2, 0) \text{ is an x-intercept}}$$

3. Leading coefficient is positive,

**Graph will be in quadrants I and III.**

4. Find the vertical asymptote. Set the denominator = 0 and solve

$$x-2 = 0$$
$$x = 2 \quad \mathbf{\text{Vertical Asymptote}}$$



5. Find the horizontal asymptote. Degree of numerator > Degree of denominator

NO **Horizontal Asymptote**

**5a. Degree of numerator is exactly one more than the degree of the denominator.**

Graph has an oblique asymptote. Divide the polynomial functions.

$$\begin{array}{r}
 x + 4 \\
 x - 2 \overline{) x^2 + 2x - 3} \\
 \underline{x^2 \pm 2x} \phantom{- 3} \\
 4x - 3 \\
 \underline{4x \pm 8} \\
 5
 \end{array}$$

$$y = x + 4 \text{ **Oblique Asymptote**}$$

Disregard the remainder.

6. Find some points in each interval of the domain.

$$\text{Domain} = (-\infty, 2) \cup (2, \infty)$$

$x$	-3	3
$f(x)$	0	12

EXAMPLE 5: Sketch the graph of a rational function that is not in lowest terms

$$f(x) = \frac{x^2 - 9}{x - 3}$$

### STEPS

Preliminaries:

a. Find the domain of  $f(x)$ .

$$\text{Domain} = (-\infty, 3) \cup (3, \infty)$$

3 is an extraneous value

b. Simplify  $f(x)$ .

$$\begin{aligned} f(x) &= \frac{(x-3)(x+3)}{x-3} \\ &= x+3 \end{aligned}$$

$$f(x) = x + 3 \quad x \neq 3 \text{ because it is not in the domain of } f(x).$$

Proceed to sketch the graph of  $f(x) = x + 3 \quad x \neq 3$

1. Find the y-intercept. Evaluate  $f(0)$

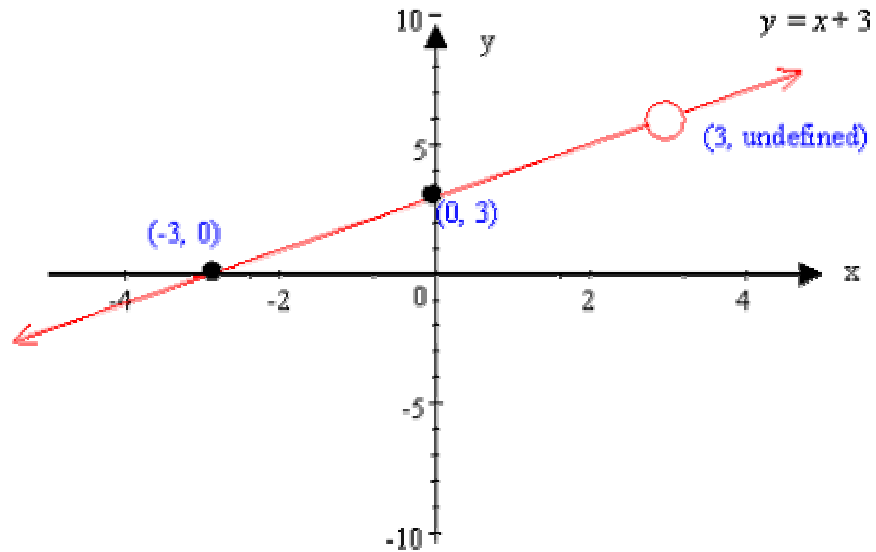
$$f(0) = 3 \quad \mathbf{(0, 3) \text{ is the y-intercept}}$$

2. Find the x-intercept. Set  $f(x) = 0$  and solve

$$x + 3 = 0 \rightarrow x = -3 \quad \mathbf{(-3, 0) \text{ is an x-intercept}}$$

3. Leading coefficient is positive,

**Graph will be in quadrants I and III.**



4. Find the vertical asymptote. Set the denominator = 0 and solve

The denominator of  $f(x)$  has been eliminated. Thus, a vertical asymptote does not exist.

**However**, a bubble is drawn to show that 3 is not in the domain of  $f(x)$  and a break occurs at that point.

5. Find the horizontal asymptote. Degree of numerator  $>$  Degree of denominator

**Does not apply.**

6. Find some points in each interval of the domain.

$$\text{Domain} = (-\infty, 3) \cup (3, \infty)$$

$x$	-3	0
$f(x)$	0	3