3.4. RATIONAL FUNCTIONS.

A rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P(x) and Q(x) are polynomial functions and $Q(x) \neq 0$.

It is important to be able to identify the domain of rational functions. The domain is the set of all real numbers excluding those x-values that make Q(x) = 0.

What happens when Q(x) = 0?



becomes undefined when the denominator equals zero.

In this lesson, we will focus on the behavior of the graphs of rational functions.

We will examine the graphs near the *x*-values that make these functions undefined.

But first. A quick review.

The following topics are useful in determining the graphical behavior of polynomial functions.

- The degree and the leading coefficient of a polynomial function.
 - What happens when the degree of a polynomial is odd? When it is even?
 - What happens when the leading coefficient is positive? When it is negative?
- Descartes' Rule of signs.
 - What happens when the number of sign variations for P(x) is odd? When it is even?
 - What happens when the number of sign variations for P(x) is greater than or equal to 2.
- *y*-intercept.

You will apply these topics and others to determine the graphical behavior of rational functions.

Let's look at the domain and the behavior of the graph of some rational functions.

 $f(x) = \frac{1}{x}$, the simplest rational function.

The domain of f(x) is the set of all real numbers except 0.

Now examine the table of ordered pairs to determine the behavior of f(x).

x-values from $-\infty$ to 0

x	x continues to approach $x \rightarrow -\infty$ $(x \rightarrow -\infty)$	- 1,000	-100	-10	-2	-1	- 0.5	-0.1	- 0.01	- 0.001	- 0.0001	$ \begin{array}{c} x \\ \text{continues} \\ \text{to} \\ \text{approach} \\ 0 \\ (x \rightarrow 0) \end{array} $	0
f(x)	f(x) continues to approach 0 (f(x) \rightarrow 0)	- 0.001	- 0.01	- 0.10	- 0.5	-1	-2	-10	-100	- 1,000	- 10,000	f(x) continues to approach $-^{\infty}$ $(f(x) \rightarrow -^{\infty})$	undefined

x-values from 0 to ∞

x	0	$ \begin{array}{c} x \text{ continues} \\ \text{to approach} \\ 0 \\ (x \rightarrow 0) \end{array} $	0.001	0.01	0.1	0.5	1	2	10	100	1,000	$\begin{array}{c} x \text{ continues} \\ \text{to} \\ \text{approach} \\ (x \rightarrow \infty) \end{array}$
f(x)	undefined	$ \begin{array}{c} f(x) \\ \text{continues} \\ \text{to} \\ \text{approach} \\ (f(x) \rightarrow \infty \\) \end{array} $	1,000	100	10	2	1	0.5	0.1	0.01	0.001	f(x) continues to approach 0 $(f(x) \rightarrow 0)$

Note:

- As x approaches zero (x → 0) from the left, f(x) decreases without bound
 As x approaches zero (x → 0) from the right, f(x) increases without bound.

Since x cannot equal zero, a break occurs and the graph of f(x) will never intersect the vertical line x = 0.

This vertical line is called a vertical asymptote.

Also note:

As x decreases or x increases in value, the f(x) approaches zero. •

We say that f(x) approaches zero as the absolute value of x increases, $(f(x) \rightarrow 0)$ as $|_{x}| \rightarrow \infty$).

The graph of f(x) gets closer and closer to the horizontal line y = 0.

This horizontal line is called a horizontal asymptote.

Unlike vertical asymptotes, the graphs of some rational functions may cross the horizontal asymptote.

Let's examine the graph.



EXAMPLE 2:
$$f(x) = \frac{2}{x+3}$$

-3

The domain of f(x) is the set of all real numbers except -3.

Now examine the table of ordered pairs to determine the behavior of f(x).

to approach 2.9999 2.999 2.99 2.9 - -

				\mathcal{A}	varue	b	11	UII	.1	ω	5			
x		x continues to approach $x^{-\infty}$ (x $\rightarrow x^{-\infty}$)	-1,000	-100	-10	- 5	-4	- 3.5	- 3.1	- 3.01	3.001	- 3.0001	x continues to approach -3 (x \rightarrow -3)	-3
fl	(x)	f(x) continues to approach 0 (f(x) \rightarrow 0)	0.00201	- 0.02062	0.28571	- 1	-2	-4	-20	-200	- 2,000	- 20,000	$f(x)$ continues to approach $-^{\infty}$ $(f(x) \rightarrow -^{\infty})$	undefined
	<i>x</i> -values from -3 to ∞													
		2	x contin	ues -	-		-	-			0	1	10 X	continues

0

1

10

to

x-values from $-\infty$ to -3

		$(x \rightarrow -3)$					2 1	L				$\operatorname{approach}_{(x \to \infty)}^{\infty}$
f(x)	undefined	f(x) continues to approach ∞ $(f(x) \rightarrow \infty$)	20,000	2,000	200	20	2 1	l	0.666667	0.5	0.153846	f(x) continues to approach 0 $(f(x) \rightarrow 0)$

Note:

- As x approaches -3 (x → -3) from the left, f(x) decreases without bound
 As x approaches -3 (x → -3) from the right, f(x) increases without bound.

Since x cannot equal -3, a break occurs and the graph of f(x) will never intersect the vertical line x = -3.

Also note:

• As x decreases or x increases in value, f(x) approaches zero.

We say that f(x) approaches zero as the absolute value of x increases $(f(x) \rightarrow 0$ as $x \mid$ ∞).

The graph of f(x) gets closer and closer to the horizontal line y = 0.

Let's examine the graph.



EXAMPLE 3:
$$f(x) = \frac{x^2 - 1}{x^2 - x - 6}$$

The domain of f(x) is the set of all real numbers except -2 and 3. *x*-values in each interval of the domain will be selected.

Now examine the table of ordered pairs to determine the behavior of f(x).

x-values from $-\infty$ to -2

x	x continues to approach $x^{-\infty}$ (x \to x^{-\infty})	-1000	-100	-10	- 5	-3	-2.9	-2.09	-2.009	-2.0009	x continues to approach -2 $(x \rightarrow -2)$	-2
f(x)	$f(x)$ continues to approach 1 $(f(x) \rightarrow 1)$	0.9990	0.9906	0.9519	1	1.3333	1.3955	7.3523	67.3472	667.3467	$f(x)$ continues to approach ∞ $(f(x) \rightarrow \infty$)	undefined
				χ	(- \	value	s fro	-2 m	to 3			
x	-2	χ	c –	1.999			-	0 1	2 2.	9 2.99	X	3

		continues		1.99	1.9	-						continues	
		to				1						to	
		approach										approach	
		-2										3	
		$(x \rightarrow -2)$										$(x \rightarrow 3)$	
		f(x)										f(x)	
		continues										continues	
		to										to	
f(x)	undefined	approach	500 320	- 50 321	- 5 327	0	1.667	0	- 0 75	-	-	approach	undefined
		_ ∞	577.520	57.521	5.521	U			0.75	13.122	137.12	_∞	
		$(f(x) \rightarrow$										$(f(x) \rightarrow$	
		- [∞])										-°°)	

x-values from 3 to $^{\circ\circ}$

x	3	x continues to approach 3 $(x \rightarrow 3)$	3.001	3.01	3.1	3.5	4	10	100	1000	x continues to approach $^{\infty}$ $(x \rightarrow ^{\infty})$
f(x)	undefined	f(x) continues to approach ∞ $(f(x) \rightarrow \infty$)	1600.88	160.88	16.8824	4.091	2.5	1.179	1.011	1.001	f(x) continues to approach 1 $(f(x) \rightarrow 1)$

Note:

- As x approaches -2 ($x \rightarrow -2$) from the left, f(x) increases without bound
- As x approaches -2 ($x \rightarrow -2$) from the right, f(x) decreases without bound.
- As x approaches 3 ($x \rightarrow 3$) from the left, f(x) decreases without bound
- As x approaches 3 ($x \rightarrow 3$) from the right, f(x) increases without bound.

Since *x* cannot equal -2 and *x* cannot equal 3, two breaks occurs and the graph of f(x) will never intersect the vertical lines x = -2 and x = 3.

Also note:

• As x decreases or x increases in value, f(x) approaches 1.

We say that f(x) approaches 1 as the absolute value of x increases, $(f(x) \rightarrow 1 \text{ as } x \rightarrow \infty)$.

The graph of f(x) gets closer and closer to the horizontal line y = 1.

Let's examine the graph.



A definition for vertical and horizontal asymptote of the graph of written in lowest terms, can be stated as follows:

The line x = a is a **vertical asymptote** if the absolute value of f(x) approaches

infinity $(f(x) \rightarrow \infty)$ as x approaches $a (x \rightarrow a)$.

The line y = b is a **horizontal asymptote** if f(x) approaches b ($f(x) \rightarrow b$) as the absolute value of x approaches infinity ($x \rightarrow \infty$)

Now, let's find out how to locate these asymptotes.

VERTICAL ASYMPTOTE

The vertical asymptote is obvious. Each excluded value of the domain is a vertical asymptote.

To determine which values are excluded, set the denominator equal to zero and solve.

HORIZONTAL ASYMPTOTE

To locate the horizontal asymptote is more interesting. Let f be a rational function

 $f(x) = \frac{a_m x^m + \dots + a_0}{b_n x^n + \dots + b_0}, \text{ written in lowest terms}$

There are three possibilities based on the <u>trichotomy property</u> (m < n, m = n, or m > n) for (m) equal the degree of the numerator and (n) equal the degree of the denominator.

CASE 1 (<i>m</i> < <i>n</i>):	The degree of the numerator is less than the degree of the denominator. The <i>x</i> -axis is the horizontal asymptote.
CASE 2 $(m = n)$:	The line $y = 0$ is the horizontal asymptote The degree of the numerator is equal to the degree of the denominator. The quotient of the leading coefficients is the horizontal asymptote.
	$y = \frac{a_m}{b}$
	The line is a horizontal
CASE 3 $(m > n)$:	The degree of the numerator is greater than the degree of the denominator.
	The graph of <i>f</i> has no horizontal

If the degree of the numerator is exactly one more than the degree of the denominator, the graph of f may have an **oblique** (slant- neither vertical or horizontal) **asymptote**.

asymptote.

An oblique asymptote can be determined by finding the quotient of the two polynomials disregarding any remainders.

EXAMPLE:
$$f(x) = \frac{x^2 + 2x - 3}{x - 2}$$

.

The graph of f(x) has no horizontal asymptote. Instead, locate an oblique (slant) asymptote.

