3.4. RATIONAL FUNCTIONS.

A rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P(x) and Q(x) are polynomial functions and Q(x) $\neq$ 0.

It is important to be able to identify the domain of rational functions. The domain is the set of all real numbers excluding those x-values that make Q(x) = 0.

What happens when Q(x) = 0?

You are correct. The rational function becomes undefined when the denominator equals zero.

In this lesson, we will focus on the behavior of the graphs of rational functions.

We will examine the graphs near the *x*-values that make these functions undefined.

But first. A quick review.

The following topics are useful in determining the graphical behavior of polynomial functions.

- The degree and the leading coefficient of a polynomial function.
  - What happens when the degree of a polynomial is odd? When it is even?
  - What happens when the leading coefficient is positive? When it is negative?
- Descartes' Rule of signs.
  - What happens when the number of sign variations for *P*(*x*) is odd? When it is even?
  - What happens when the number of sign variations for *P*(*x*) is greater than or equal to 2.
- *y*-intercept.

You will apply these topics and others to determine the graphical behavior of rational functions.

Let's look at the domain and the behavior of the graph of some rational functions.

EXAMPLE 1: $f(x) = \frac{1}{x}$, the simplest rational function.

The domain of $f(x)$ is the set of all real numbers except 0.

Now examine the table of ordered pairs to determine the behavior of $f(x)$.

x-values from $-\infty$ to 0

x	x continues to approach $-\infty$ ($x \to -\infty$)	-1,000	-100	-10	-2	-1	-0.5	-0.1	-0.01	-0.001	-0.0001	x continues to approach 0 ($x \to 0$)	0
$f(x)$	$f(x)$ continues to approach 0 ($f(x) \to 0$)	-0.001	-0.01	-0.10	-0.5	-1	-2	-10	-100	-1,000	-10,000	$f(x)$ continues to approach $-\infty$ ($f(x) \to -\infty$)	undefined

x-values from 0 to ∞

x	0	x continues to approach 0 ($x \to 0$)	0.001	0.01	0.1	0.5	1	2	10	100	1,000	x continues to approach ∞ ($x \to \infty$)
$f(x)$	undefined	$f(x)$ continues to approach ∞ ($f(x) \to \infty$)	1,000	100	10	2	1	0.5	0.1	0.01	0.001	$f(x)$ continues to approach 0 ($f(x) \to 0$)

Note:

- As x approaches zero ($x \to 0$) from the left, $f(x)$ decreases without bound
- As x approaches zero ($x \to 0$) from the right, $f(x)$ increases without bound.

Since x cannot equal zero, a break occurs and the graph of $f(x)$ will never intersect the vertical line $x = 0$.

This vertical line is called a vertical asymptote.

Also note:

- As x decreases or x increases in value, the $f(x)$ approaches zero.

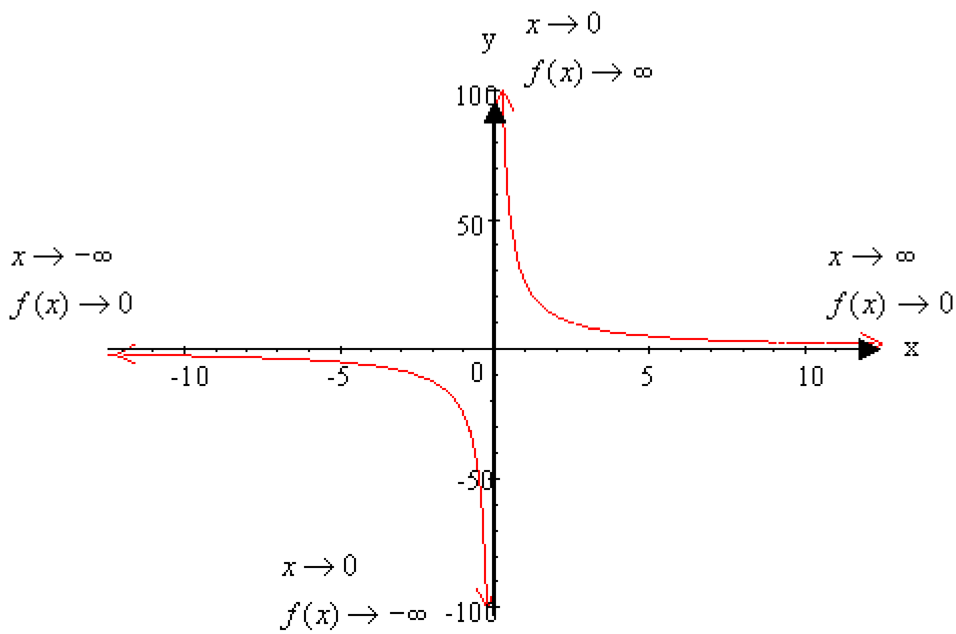
  We say that $f(x)$ approaches zero as the absolute value of x increases, ($f(x) \to 0$ as $|x| \to \infty$).

The graph of $f(x)$ gets closer and closer to the horizontal line $y = 0$.

This horizontal line is called a horizontal asymptote.

Unlike vertical asymptotes, the graphs of some rational functions may cross the horizontal asymptote.

Let's examine the graph.



The graph of $f(x)$ is a hyperbola.

EXAMPLE 2: $f(x) = \dfrac{2}{x+3}$

The domain of $f(x)$ is the set of all real numbers except -3.

Now examine the table of ordered pairs to determine the behavior of $f(x)$.

x-values from $-\infty$ to -3

x	x continues to approach $-\infty$ ($x \to -\infty$)	-1,000	-100	-10	-5	-4	-3.5	-3.1	-3.01	-3.001	-3.0001	x continues to approach -3 ($x \to -3$)	-3
$f(x)$	$f(x)$ continues to approach 0 ($f(x) \to 0$)	-0.00201	-0.02062	-0.28571	-1	-2	-4	-20	-200	-2,000	-20,000	$f(x)$ continues to approach $-\infty$ ($f(x) \to -\infty$)	undefined

x-values from -3 to ∞

x	-3	x continues to approach	-2.9999	-2.999	-2.99	-2.9	-	-	0	1	10	x continues to

		-3 ($x \to$ -3)					2	1				approach ∞ ($x \to \infty$)
$f(x)$	undefined	$f(x)$ continues to approach ∞ ($f(x) \to \infty$)	20,000	2,000	200	20	2	1	0.666667	0.5	0.153846	$f(x)$ continues to approach 0 ($f(x) \to$ 0)

Note:

- As x approaches -3 ($x \to$ -3) from the left, $f(x)$ decreases without bound
- As x approaches -3 ($x \to$ -3) from the right, $f(x)$ increases without bound.

Since x cannot equal -3, a break occurs and the graph of $f(x)$ will never intersect the vertical line x = -3.

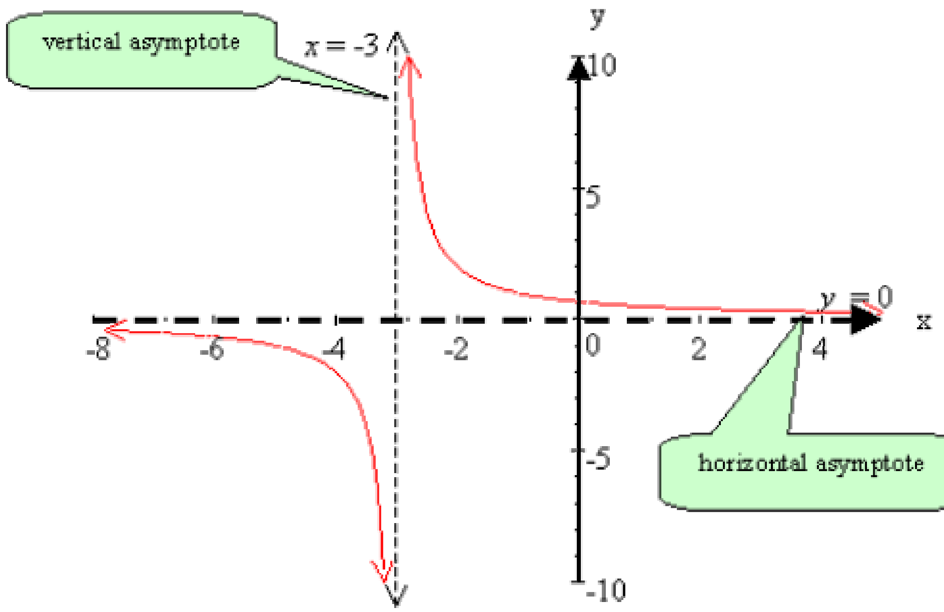
Also note:

- As x decreases or x increases in value, $f(x)$ approaches zero.

  We say that $f(x)$ approaches zero as the absolute value of x increases ($f(x) \to$ 0 as $|x|$ ∞).

The graph of $f(x)$ gets closer and closer to the horizontal line y = 0.

Let's examine the graph.



EXAMPLE 3: $f(x) = \dfrac{x^2 - 1}{x^2 - x - 6}$

The domain of *f*(*x*) is the set of all real numbers except -2 and 3. *x*-values in each interval of the domain will be selected.

Now examine the table of ordered pairs to determine the behavior of *f*(*x*).

x-values from -∞ to -2

x	*x* continues to approach -∞ ($x \to -\infty$)	-1000	-100	-10	-5	-3	-2.9	-2.09	-2.009	-2.0009	*x* continues to approach -2 ($x \to -2$)	-2
f(*x*)	$f(x)$ continues to approach 1 ($f(x) \to 1$)	0.9990	0.9906	0.9519	1	1.3333	1.3955	7.3523	67.3472	667.3467	$f(x)$ continues to approach ∞ ($f(x) \to \infty$)	undefined

x-values from -2 to 3

x	-2	*x*	-1.999	-	-		0	1	2	2.9	2.99	*x*	3

		continues to approach -2 ($x \to -2$)		1.99	1.9	-1						continues to approach 3 ($x \to 3$)	
$f(x)$	undefined	$f(x)$ continues to approach $-\infty$ ($f(x) \to -\infty$)	-599.320	-59.321	-5.327	0	1.667	0	-0.75	-15.122	-159.12	$f(x)$ continues to approach $-\infty$ ($f(x) \to -\infty$)	undefined

x-values from 3 to ∞

x	3	x continues to approach 3 ($x \to 3$)	3.001	3.01	3.1	3.5	4	10	100	1000	x continues to approach ∞ ($x \to \infty$)
$f(x)$	undefined	$f(x)$ continues to approach ∞ ($f(x) \to \infty$)	1600.88	160.88	16.8824	4.091	2.5	1.179	1.011	1.001	$f(x)$ continues to approach 1 ($f(x) \to 1$)

Note:

- As x approaches -2 ($x \to -2$) from the left, $f(x)$ increases without bound
- As x approaches -2 ($x \to -2$) from the right, $f(x)$ decreases without bound.
- As x approaches 3 ($x \to 3$) from the left, $f(x)$ decreases without bound
- As x approaches 3 ($x \to 3$) from the right, $f(x)$ increases without bound.

Since x cannot equal -2 and x cannot equal 3, two breaks occurs and the graph of $f(x)$ will never intersect the vertical lines $x = -2$ and $x = 3$.

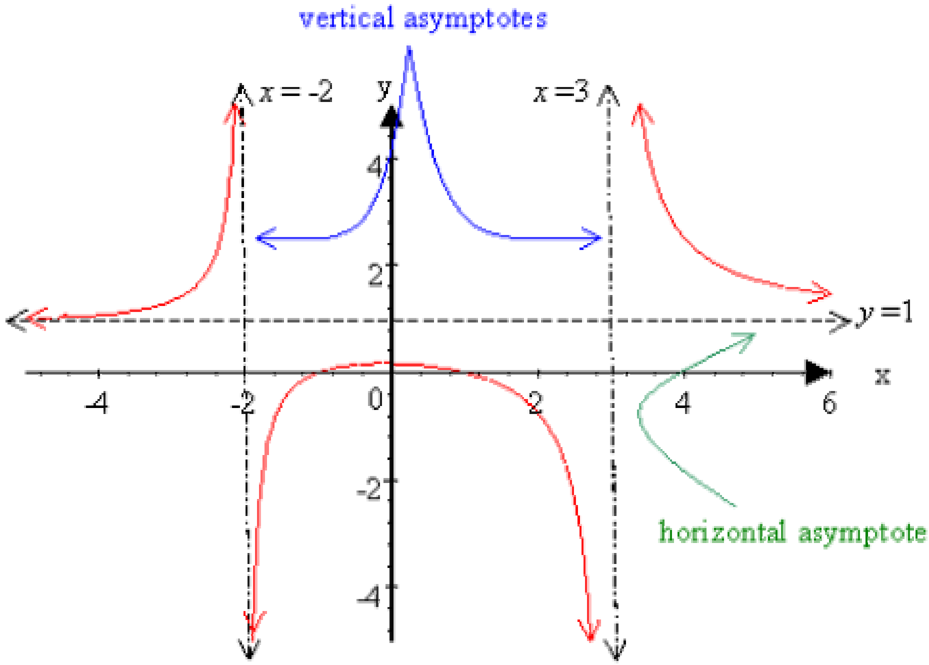
Also note:

- As x decreases or x increases in value, $f(x)$ approaches 1.

We say that $f(x)$ approaches 1 as the absolute value of x increases, ($f(x) \to 1$ as $x \to \infty$).

The graph of $f(x)$ gets closer and closer to the horizontal line $y = 1$.

Let's examine the graph.



A definition for vertical and horizontal asymptote of the graph of $f(x) = \dfrac{P(x)}{Q(x)}$, written in lowest terms, can be stated as follows:

The line $x = a$ is a **vertical asymptote** if the absolute value of $f(x)$ approaches infinity ($f(x) \to \infty$) as x approaches a ($x \to a$).

The line $y = b$ is a **horizontal asymptote** if $f(x)$ approaches b ($f(x) \to b$) as the absolute value of x approaches infinity ($x \to \infty$)

Now, let's find out how to locate these asymptotes.

VERTICAL ASYMPTOTE

The vertical asymptote is obvious. Each excluded value of the domain is a vertical asymptote.

To determine which values are excluded, set the denominator equal to zero and solve.

HORIZONTAL ASYMPTOTE

To locate the horizontal asymptote is more interesting. Let f be a rational function given by $f(x) = \dfrac{a_m x^m + . \; . \; . + a_0}{b_n x^n + . \; . \; . + b_0}$, written in lowest terms

There are three possibilities based on the trichotomy property ($m < n$, $m = n$, or $m > n$) for (m) equal the degree of the numerator and (n) equal the degree of the denominator .

CASE 1 ($m < n$): The degree of the numerator is less than the degree of the denominator. The x-axis is the horizontal asymptote.

The line $y = 0$ is the **horizontal asymptote .**

CASE 2 ($m = n$) : The degree of the numerator is equal to the degree of the denominator. The quotient of the leading coefficients is the horizontal asymptote.

The line $y = \dfrac{a_m}{b_n}$ is a **horizontal asymptote .**

CASE 3 ($m > n$): The degree of the numerator is greater than the degree of the denominator.

The graph of f has no **horizontal asymptote .**

If the degree of the numerator is exactly one more than the degree of the denominator, the graph of f may have an **oblique** (slant- neither vertical or horizontal) **asymptote**.

An oblique asymptote can be determined by finding the quotient of the two polynomials disregarding any remainders.

EXAMPLE: $f(x) = \dfrac{x^2 + 2x - 3}{x - 2}$

The graph of $f(x)$ has no horizontal asymptote. Instead, locate an oblique (slant) asymptote.

$$\begin{array}{r} x + 4 \\ x - 2 \overline{\smash{\big)} x^2 + 2x - 3} \\ \underline{x^2 \pm 2x} \\ 4x - 3 \\ \underline{4x \pm 8} \\ \cancel{5} \end{array} \implies$$

$y = x + 4$ is the oblique asymptote.

Disregard the remainder