

4.1. Exponents

Repeated multiplication of a real number by itself can be written in **exponential form**.

Example: *Repeated Multiplication:* $2 \cdot 2 \cdot 2$

Exponential Form: 2^3

Example: *Repeated Multiplication:* $b \cdot b \cdot b \cdot b$

Exponential Form: b^4

Definition of Exponential Notation:

Let a be a real number, a variable or an algebraic expression, and let n be a positive integer.

Then $a^n = a \cdot a \cdot a \cdot \dots \cdot a$ (**n factors of a**)

where n is the **exponent** and a is the **base**.

The expression a^n is read as " a to the **n th power**" or simply a to the n th".

Note: *Exponents are used in "exponential notation" which is a short way of writing products with the same factor repeating a certain number of times.*

Note: *The base could be any number but only counting numbers can be used as exponents.*

Example: $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$ where the base is 2 and the exponent is 5.

Example: $(-3)^4 = (-3) \cdot (-3) \cdot (-3) \cdot (-3) = 81$ where the base is (-3) and the exponent is 4.

Example: $-3^4 = -(3 \cdot 3 \cdot 3 \cdot 3) = -81$ where the base is 3 and the exponent is 4.

Note: *Notice that $-$ is not a part of the base in the last example.*

Work with exponents can be simplified by using the **rules (properties) of exponents**.

Properties of Exponents: For all integers m and n and all real numbers a and b then:

1) Product Rule: $a^m \cdot a^n = a^{m+n}$

Example: $y^4 \cdot y^7 = y^{4+7} = y^{11}$

Note: The products of exponential expressions with the same base are found by adding exponents.

2) Quotient Rule:

$$\frac{a^m}{a^n} = a^{m-n} \quad \text{provided } a \neq 0 \text{ (since division by zero is undefined).}$$

Example:

$$\frac{x^7}{x^4} = x^{7-4} = x^3 \quad \text{provided } x \neq 0.$$

Example:

$$\frac{x^4}{x^7} = \frac{1}{x^{7-4}} = \frac{1}{x^3} \quad \text{provided } x \neq 0.$$

Note: The quotients of exponential expressions are found by subtracting lesser exponent from greater exponent.

3) Zero Exponent: $a^0 = 1$ provided $a \neq 0$.

Example: $2^0 = 1$ or $b^0 = 1$ provided $b \neq 0$.

Note: Any exponential expression raised to the zero power is one. The symbol 0^0 is undefined.

4) Negative Exponent:

$$a^{-n} = \frac{1}{a^n} \quad \text{provided } a \neq 0.$$

Example:

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

Note: Negative exponents lead to reciprocals.

5) Power Rules:

$$\begin{aligned}(a^m)^n &= a^{mn} \\ (ab)^m &= a^m b^m \\ \left(\frac{a}{b}\right)^m &= \frac{a^m}{b^m} \quad \text{provided } b \neq 0.\end{aligned}$$

Examples:

$$\begin{aligned}(2^3)^2 &= 2^{3 \cdot 2} = 2^6 = 64 \\ (2xy)^3 &= 2^3 x^3 y^3 = 8x^3 y^3 \\ \left(\frac{x}{3}\right)^2 &= \frac{x^2}{3^2} = \frac{x^2}{9}\end{aligned}$$

Note: In order to remove parentheses a product of exponents must occur.

Example:

$$\left(\frac{a^2 b^{-3}}{a^4 b^2}\right)^2 = \frac{a^{2 \cdot 2} b^{-3 \cdot 2}}{a^{4 \cdot 2} b^{2 \cdot 2}} = \frac{a^4 b^{-6}}{a^8 b^4} = \frac{1}{a^{8-4} b^{4-(-6)}} = \frac{1}{a^4 b^{10}}$$

6) Special Rules:

$$\begin{aligned}a^{-n} &= \left(\frac{1}{a}\right)^n \\ \frac{1}{a^{-n}} &= a^n \\ \left(\frac{a}{b}\right)^{-n} &= \left(\frac{b}{a}\right)^n \quad \text{provided } a, b \neq 0.\end{aligned}$$

Examples:

$$2^{-3} = \left(\frac{1}{2}\right)^3 = \frac{1^3}{2^3} = \frac{1}{8}$$

$$\frac{1}{3^{-2}} = 3^2 = 9$$

$$\left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^3 = \frac{3^3}{2^3} = \frac{27}{8}$$

Note: Negative exponents are being removed first followed by parentheses.

So far you have dealt with only **algebraic functions**, which included polynomial and rational functions. You will now study one type of nonalgebraic function called an **exponential function**.

Definition of Exponential Function:

The **exponential function f with base a** is denoted by $f(x) = a^x$ where $a > 0$, $a \neq 1$, and x is any real number.

Example: $f(x) = 2^x$ is an exponential function because $a = 2$ which verifies $a > 0$ and $a \neq 1$.

Note: a is the base and the base must be a positive real number with the exception of the number one.

Example: $f(x) = x^2$ is **not** an exponential function because the base must be a specified positive real number with the exception of one. This is an example of an algebraic function.

Note: The base $a = 1$ is excluded because it yields $f(x) = 1^x = 1$. This is a constant function, not an exponential function.