4.1. Exponents

Repeated multiplication of a real number by itself can be written in exponential form.

Example: *Repeated Multiplication:* $2 \cdot 2 \cdot 2$

Exponential Form: 2^3

Example: *Repeated Multiplication:* $b \cdot b \cdot b \cdot b$

Exponential Form: b^4

Definition of Exponential Notation:

Let *a* be a real number, a variable or an algebraic expression, and let *n* be a positive integer.

Then $a^n = a \cdot a \cdot a \cdot a \ldots \cdot a$ (***n factors of a***)

where *n* is the exponent and *a* is the base.

The expression a^n is read as "*a* to the *n*th power" or simply *a* to the *n*th".

Note: *Exponents are used in "exponential notation" which is a short way of writing products with the same factor repeating a certain number of times.*

Note: *The base could be any number but only counting numbers can be used as exponents*.

Example: $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$ where the base is 2 and the exponent is 5.

Example: $(-3)^4 = (-3) \cdot (-3) \cdot (-3) \cdot (-3) = 81$ where the base is (-3) and the exponent is 4.

Example: $- 3^4 = - (3 \cdot 3 \cdot 3 \cdot 3) = - 81$ where the base is 3 and the exponent is 4.

Note: *Notice that - is not a part of the base in the last example*.

Work with exponents can be simplified by using the rules (properties) of exponents.

Properties of Exponents: For all integers *m* and *n* and all real numbers *a* and *b* then:

1) Product Rule: $a^m \cdot a^n = a^{m+n}$

Example: $y^4 \cdot y^7 = y^{4+7} = y^{11}$

Note: *The products of exponential expressions with the same base are found by adding exponents.*

2) Quotient Rule:

$$\frac{a^m}{a^n} = a^{m-n}$$ provided $a \neq 0$ (since division by zero is undefined).

Example:

$$\frac{x^7}{x^4} = x^{7-4} = x^3$$ provided $x \neq 0$.

Example:

$$\frac{x^4}{x^7} = \frac{1}{x^{7-4}} = \frac{1}{x^3}$$ provided $x \neq 0$.

Note: *The quotients of exponential expressions are found by subtracting lesser exponent from greater exponent.*

3) Zero Exponent: $a^0 = 1$ provided $a \neq 0$.

Example: $2^0 = 1$ or $b^0 = 1$ provided $b \neq 0$.

Note: *Any exponential expression raised to the zero power is one. The symbol 0^0 is undefined.*

4) Negative Exponent:

$$a^{-n} = \frac{1}{a^n}$$ **provided $a \neq 0$.**

Example:

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

Note: *Negative exponents lead to reciprocals.*

5) Power Rules:

$$\left(a^m\right)^n = a^{mn}$$
$$(ab)^m = a^m b^m$$
$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$ provided b ≠ 0.

Examples:

$$\left(2^3\right)^2 = 2^{3 \cdot 2} = 2^6 = 64$$
$$(2xy)^3 = 2^3 x^3 y^3 = 8x^3 y^3$$
$$\left(\frac{x}{3}\right)^2 = \frac{x^2}{3^2} = \frac{x^2}{9}$$

Note: *In order to remove parentheses a product of exponents must occur.*

Example:

$$\left(\frac{a^2 b^{-3}}{a^4 b^2}\right)^2 = \frac{a^{2 \cdot 2} b^{-3 \cdot 2}}{a^{4 \cdot 2} b^{2 \cdot 2}} = \frac{a^4 b^{-6}}{a^8 b^4} = \frac{1}{a^{8-4} b^{4--6}} = \frac{1}{a^4 b^{10}}$$

6) Special Rules:

$$a^{-n} = \left(\frac{1}{a}\right)^n$$
$$\frac{1}{a^{-n}} = a^n$$
$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$ provided a, b ≠ 0.

Examples:

$$2^{-3} = \left(\frac{1}{2}\right)^3 = \frac{1^3}{2^3} = \frac{1}{8}$$

$$\frac{1}{3^{-2}} = 3^2 = 9$$

$$\left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^3 = \frac{3^3}{2^3} = \frac{27}{8}$$

Note: *Negative exponents are being removed first followed by parentheses.*

So far you have dealt with only algebraic functions, which included polynomial and rational functions. You will now study one type of nonalgebraic function called an exponential function.

Definition of Exponential Function:

> The exponential function f with base a is denoted by $f(x) = a^x$ where $a > 0$, $a \neq 1$, and x is any real number.

Example: $f(x) = 2^x$ is an exponential function because $a = 2$ which verifies $a > 0$ and $a \neq 1$.

Note: *a is the base and the base must be a positive real number with the exception of the number one.*

Example: $f(x) = x^2$ is **not** an exponential function because the base must be a specified positive real number with the exception of one. This is an example of an algebraic function.

Note: *The base $a = 1$ is excluded because it yields $f(x) = 1^x = 1$. This is a constant function, not an exponential function.*