

5.1. Linear equations and their solution sets

General equations and their solution sets

An equation by its nature is not a declarative statement but rather a question and for this reason we cannot say that it is true or false. Each equation has variables which do not have fixed values and they are called unknowns. Logically, writing an equation we ask a question for which values of the variables the equation becomes a true equality. More precisely, an equation is solved when we know all the values of the variables which, after substitution to the equation change it into a true equality.

5.1.1. DEFINITION.

The solution set of an equation is the set of all numbers (the ordered pairs of numbers, the ordered triples of numbers, the ordered n -tuples of numbers, etc.) such that if we substitute them for the variables we obtain a true equality.

5.1.2. EXAMPLE.

Does the number 2 belong to the solution set of the equation $3x + 2 = 5$? Of course, the correct answer is NO. The substitution gives us $3 \cdot 2 + 5 = 11 \neq 5$.

5.1.3. EXAMPLE.

Does the ordered pair of numbers $(2, 3)$ belong to the solution set of the equation $x^2 + y^2 = 13$? Of course, the answer is YES. The substitution gives us $2^2 + 3^2 = 4 + 9 = 13$.

5.1.4. EXAMPLE.

Does the ordered triple of numbers $(2, 3, 4)$ belong to the solution set of the equation $x^2 + y^2 + z = 15$? Of course, the answer is NO. The substitution gives us $2^2 + 3^2 + 4 = 4 + 9 + 4 = 17 \neq 15$.

Linear equations in one variable

Linear equations in one variable are of the form

$$ax + b = 0$$

(as a specific example we use $3x + 15 = 0$). In the above equations x is a variable, b (in particular 15) is a constant numbers, and a (in particular 3) is a coefficient (constant number in the front of the unknown x). To solve the equation for x , we isolate the unknown x , by moving all the constant numbers to the right side

$$ax = -b$$

and then by dividing the both sides of the equation by the coefficient

$$x = -\frac{b}{a}.$$

The last is possible to do only if the coefficient a is not 0. In this case we obtain $x = -\frac{b}{a}$. In the specific example we have

$$3x + 15 = 0$$

$$3x = -15$$

$$x = -\frac{15}{3}$$

and we finally obtain $x = -5$.

From the method of solving the equations that we presented above we obtain that there are only three possibilities for the solution set of the equation of the form $ax + b = 0$:

- 1) the set $\{-\frac{b}{a}\}$, if $a \neq 0$,
- 2) the set $(-\infty, \infty)$ (i.e. any number is a solution), if $a = 0$ and $b = 0$,
- 3) the empty set denoted by \emptyset (i.e. there is no solution) if $a = 0$ and $b \neq 0$.

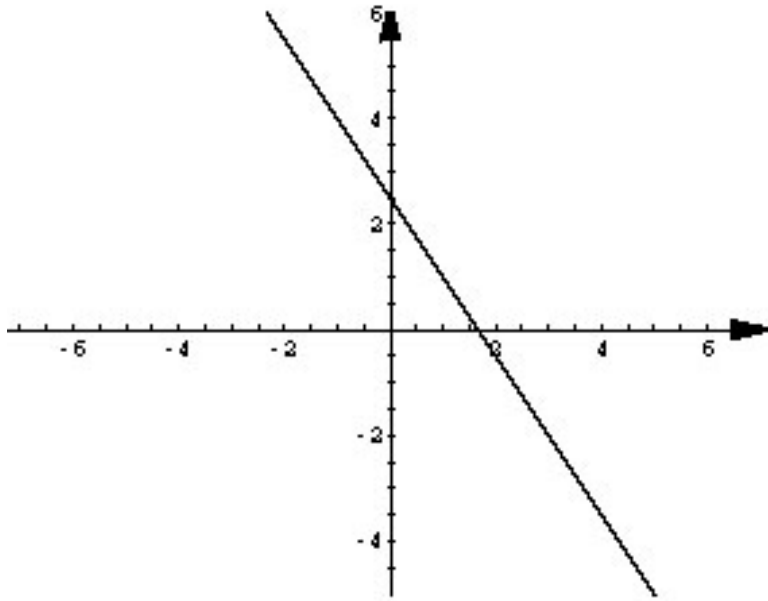
5.1.5. EXAMPLE. The solution set of the equation $3x + 2 = 5$ is $\{1\}$.

5.1.6. EXAMPLE. The solution set of the equation $3x + 2 = 3x + 2$ is $(-\infty, \infty)$.

5.1.7. EXAMPLE. The solution set of the equation $3x + 2 = 3x + 5$ is \emptyset .

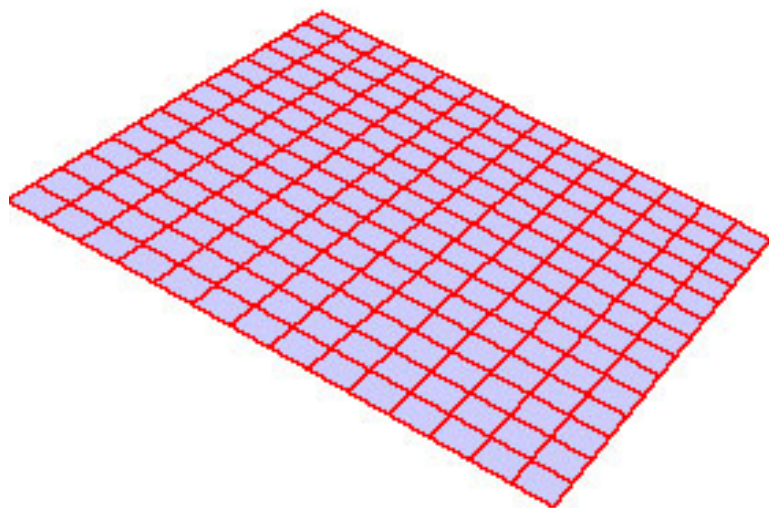
Linear equation in several variables

An equation in one variable have either $\{-b/a\}$, or $(-\infty, \infty)$, or \emptyset as the solution set. What happens if we increase the number of variables like for instance in the equation $3x + 2y = 5$? Now, a single solution is a value for the unknown x and a value for the unknown y . It generates ordered pairs of numbers like $(1, 1)$, $(5, -5)$, $(15, -20)$, and more. The solution set is infinite and the best way to describe it is graphing.



As you can see above the graph of $3x + 2y = 5$ is a straight line and it explains why equations of this type is called linear.

If we have three variables like in $3x + 2y + z = 15$ a single solution is a value for the unknown x , a value for the unknown y , and a value for the unknown z . It generates ordered triples of numbers. For this reason, the solution set can be interpreted in the three-dimensional space. Graphically, the solution set turns out to be a plane in three dimensional space.

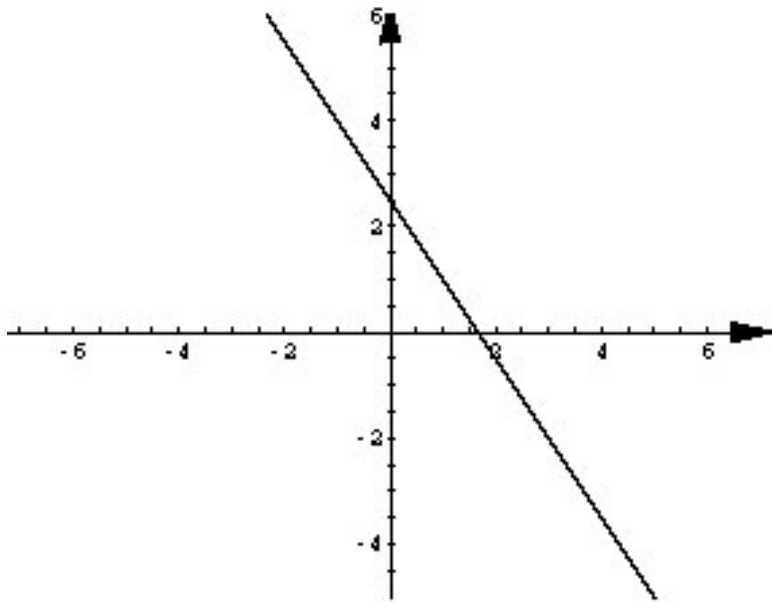


Graphs of solution sets of linear equations

5.1.9. EXAMPLE. The solution set of the equation $3x+2=5$ is $\{1\}$.



5.1.10. EXAMPLE. The solution set of the equation $3x+2y=5$ is a line in plane.



5.1.11. EXAMPLE. The solution set of the equation $3x+2y+z=0$ is a plane in the three-dimensional space.

