5.2. Systems of linear equations and their solution sets

Solution sets of systems of equations as intersections of sets

Any collection of two or more equations is called a system of equations. The solution set of a system of equations is the set of all numbers (pairs of numbers, triples of numbers,n-tuples of numbers etc.)that satisfy each equation of the system. Logically, it means that they solve the first equation and they solve the second equation and they solve the third equation and so on. For this reason, any solution of the system belongs to the solution set of the first equation and belongs to the solution set of the second equation and belongs to the solution set of the third equation and so on. So it belongs to the intersection (the common part) of the solution sets of all equations in the system. We have concluded that: the solution set of a system of equations is equal to the intersection of solution sets of all the equation in the system.

Systems of linear equations in two variables and their solution sets

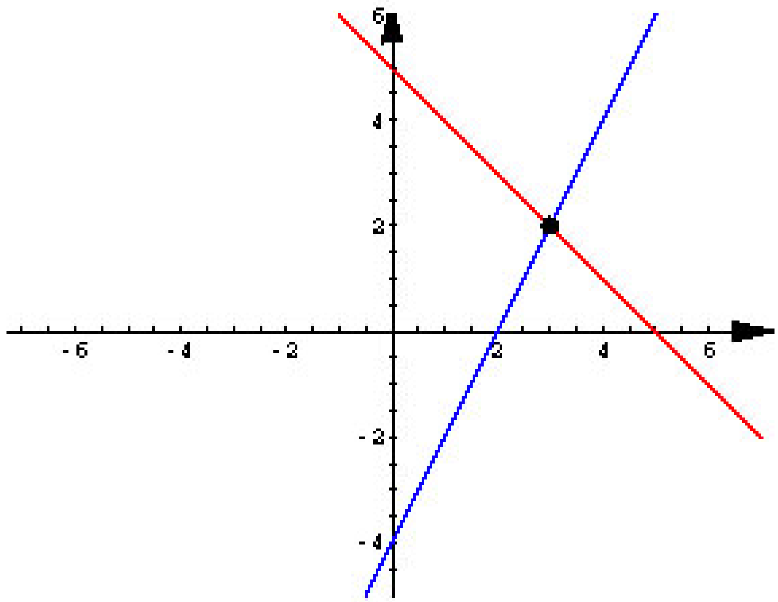
The solution set of a single linear equation in two variables is a straight line in a coordinate plane. Hence the solution sets of systems of two linear equations in two variables is the intersection of two lines. Two lines in a plane either intersect at exactly one point or they are parallel. A system which corresponds to a pair of lines intersecting at exactly one point is called *independent* and it has exactly one solution. If the two lines corresponding to the equations are parallel the system is called *inconsistent* and it has no solution. The third possibility is that the equations have the same line as their graphs. In this case the system is called *dependent* and it has infinitely many solutions. We illustrate the above three types of systems of linear equations in the examples below.

5.2.1. EXAMPLE (Independent System).

Let us consider the following linear system

$$\begin{array}{rcrcr} x & + & y & = & 5 \\ 2x & - & y & = & 4 \end{array}.$$

The solution sets of each of the equations in the system is a line. The two lines are shown below.



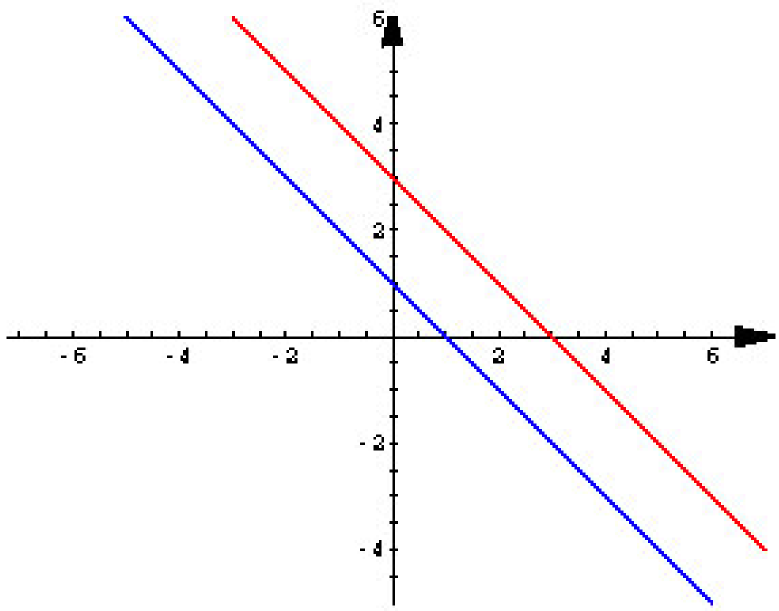
From the graph we can see that the lines intersect at the point $(3, 2)$. So the solution set for the system is $\{(3, 2)\}$.

5.2.2. EXAMPLE (Inconsistent System).

Let us consider the system

$$\begin{array}{ccccc} x & + & y & = & 1 \\ x & + & y & = & 3 \end{array}.$$

The solution sets of each of the equations in the system is a line. The two lines are shown below.



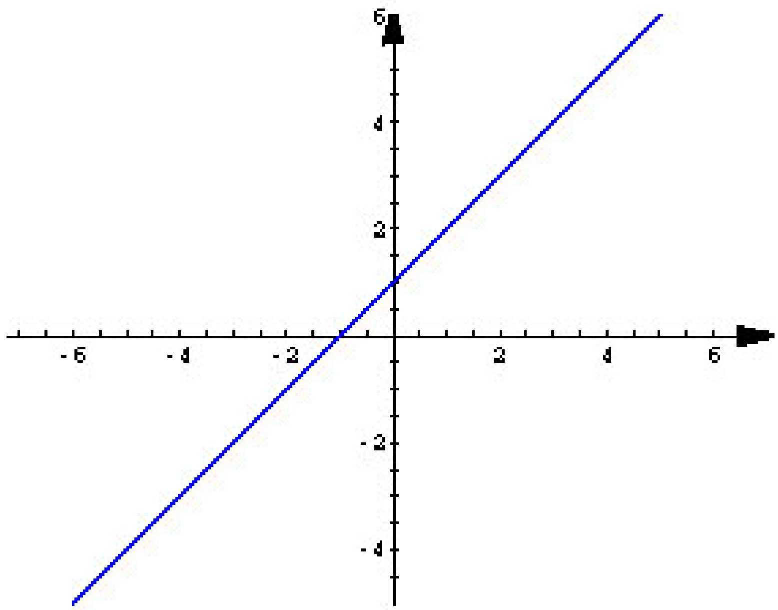
As you can see in the graph the lines are parallel and do not intersect. It means that the solution set for the system is empty (no solution).

5.2.3. EXAMPLE (Dependent System).

Let us consider the system

$$\begin{array}{rcrcr} x & - & y & = & -1 \\ 2x & - & y & = & -2 \end{array}.$$

The solution sets of each of the equations in the system is the same line which is shown below.



This time we see only one line. Which means that the graph of the first equation is on the top of the graph of the other. They are identically the same so their intersection is the line itself. It tells us that the solution set for the system is exactly the same line. The system has infinitely many solutions.

Systems of linear equation in three unknowns and their solution sets

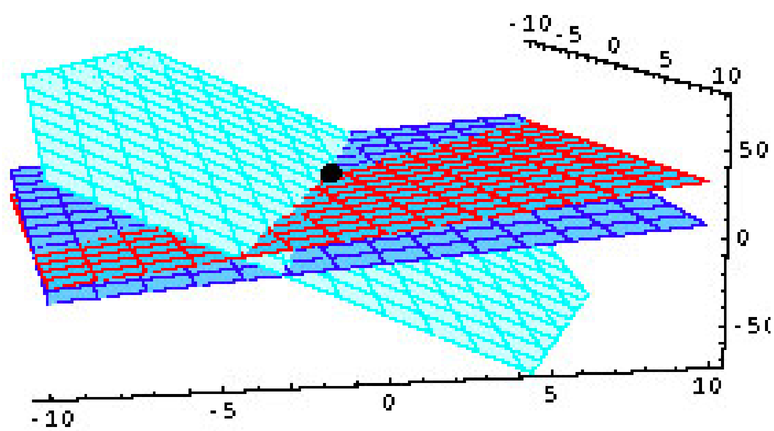
The solution sets of single linear equation in three unknowns are planes in a coordinate space. Hence the solution sets of systems of three linear equations in three unknowns is the intersection of three planes. The examples below illustrate some of the possible situation for three planes in space.

5.2.4. EXAMPLE (Independent System).

Let us consider the system

$$\begin{array}{rcrcrcr} 4x & + & 8y & + & z & = & 2 \\ x & - & 7y & - & 3z & = & -14 \\ 2x & - & 3y & + & 2z & = & 3 \end{array}$$

The graph of each individual equation is a plane and we show these planes below.



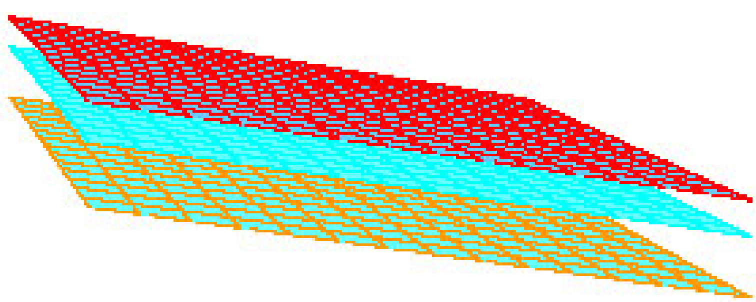
The graph shows that the planes intersect at the point $(-3, 1, 6)$. So the solution set for the system is the set $\{(-3, 1, 6)\}$.

5.2.5. EXAMPLE (Inconsistent System).

Let us consider the system

$$\begin{array}{ccccccc} x & + & 2y & + & z & = & 30 \\ x & + & 2y & + & z & = & 17 \\ x & + & 2y & + & z & = & 5 \end{array}$$

The graph of each individual equation is a plane and we show these planes below.



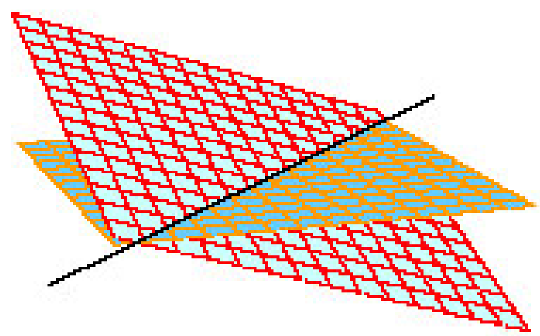
The planes are parallel and do not have common point. It means that the solution set for the system is empty and the system is inconsistent.

5.2.6. EXAMPLE (Dependent System).

Let us consider the system

$$\begin{array}{rcrcrcr} 2x & + & 4y & + & 2z & = & 8 \\ x & + & 2y & + & z & = & 4 \\ 3x & - & y & + & z & = & -9 \end{array}$$

The graph of each individual equation is a plane and we show these planes below.



Two planes are one on top of the other and they intersect the third plane along a line. So the solution set for the system is the line shown in the graph. The system has infinitely many solutions so it is dependent.