5.4. The back-substitution method

Easy to solve systems of linear equations

General approach to solving equations or system of equations is changing them into easy to solve forms. Following this approach we need to find and understand easy to solve forms for linear systems. As you can see there is no much to do to solve the system in the next example.

5.4.1. EXAMPLE. Let us consider the following system

$$
\begin{array}{rcl}
7x & = & 21 \\
-2y & = & 2 \\
3z & = & 33
\end{array}
.
$$

A system of this form is called diagonal and it is easy to solve. Dividing the both sides of the first equation by 7 give us $x = 3$. Dividing the second equation by -2 gives $y = -1$ and, finally, dividing the third equation by 3 gives $z = 11$. What makes the above system easy to solve is the fact that the variables are separated. Exactly one of them appears in each equation.

The back-substitution method

There are other than diagonal systems which can be also solved in an easy way.

5.4.2. EXAMPLE.
Let us consider the following system

$$
\begin{array}{rcl}
2x & - & 3y & = & 1 \\
& & 2y & = & 2
\end{array}
.
$$

A system of this form is called upper-triangular and it is also easy to solve. It is easy to find the value of y. Dividing the both sides of the last equation by 2 gives $y = 1$. How to find the value for x? The simplest idea is substituting 1 for y in the first equation. It gives us $2x - 3 = 1$ and $2x = 4$. Now dividing by 2 gives $x = 2$.

The method that we used to solve the above system is called *back-substitution* and it works for many linear systems.

5.4.3. EXAMPLE. Let us consider another upper-triangular system

$$
\begin{array}{rcrcrcl}
7x & + & 2y & + & 2z & = & 21 \\
   & - & 2y & + & 3z & = & 1 \\
   &   &   &   & 4z & = & 12
\end{array}
\;.
$$

We start with solving the last equation and it gives us $z = 3$. Then the substitution 3 for z in the second equation gives $-2y + 9 = 1$ which is the same as $-2y = -8$ and gives us $y = 4$. Now the substitution 3 for z and 4 for y in the first equation gives $7x + 8 + 6 = 21$ which is the same as $7x = 7$ and gives us $x = 1$.

Dependent linear systems solvable by the back-substitution method

It is possible that simplifying a linear system we "loose" one of the equation because it becomes true equality $0 = 0$. You will see more examples like later. It means that the original system was dependent (please look at the examples in Lesson 6.2.).

5.4.4. EXAMPLE.

$$
\begin{array}{rcrcrcl}
5x & + & 2y & + & z & = & 22 \\
   & - & 3y & + & 6z & = & 12 \\
   &   &   &   & 0 & = & 0
\end{array}
$$

In the above system the third equation is irrelevant. Thus we have only two equations and for this reason the solution set is infinite (it is a line in space). In the case like that solving means describing this infinite set using simple equations. The way to do it is "reducing" the number of unknowns by expressing some of them (called lead variables) by the others (called free variables). We need to decide how many free variables we have. In the case of systems in three unknowns when we are left with two equations we have one free variable. We can pick z as our free variable and express y and x in terms of z using the back-substitution method. We start with second equation and we move the term $6z$ to the right side of the equation which gives $-3y = -6z + 12$. Now, in order to isolate y, we divide by -3 and obtain $y = 2z - 4$. In order to express x in terms of z we substitute $2z - 4$ for y in the first equation. It gives $5x + 2(2z - 4) + z = 22$. Removing the parentheses and combining the like terms brings $5x + 5z - 8 = 22$. Now we move $5z - 8$ to the right side obtaining $5x = -5z + 30$ and later $x = -z + 6$. We have found that the solution set is the line in a coordinate space which can be described by $\{(-z + 6, 2z - 4, z) \ : \ z \text{ is any real number}\}$.