

## 5.5. Solving linear systems by the elimination method

### *Equivalent systems*

The major technique of solving systems of equations is changing the original problem into another one which is of an easier to solve form. In this procedure we need to be sure that the new system has the same solution set. No solution is added and no solution is lost. It leads to the concept of equivalent systems.

5.5.1. DEFINITION. We say that two systems are equivalent if and only if they have equal solution sets.

5.5.2. EXAMPLE.

Let us consider the system

$$\begin{array}{rcl} 7x + 2y + 2z & = & 21 \\ -2y + 3z & = & 1 \\ 4z & = & 12 \end{array}$$

By the back-substitution we can find that the solution set of the above system is  $\{(1, 4, 3)\}$ .

Now let us look at another system which is not exactly the same as the original one.

$$\begin{array}{rcl} 21x + 6y + 6z & = & 63 \\ -4y + 6z & = & 12 \\ 2z & = & 6 \end{array}$$

Applying the back-substitution again we find that the solution set of this system is also  $\{(1, 4, 3)\}$ . It means that the second system is equivalent to the original system.

Now let us consider the third system

$$\begin{array}{rcl} 7x + 2y + 2z & = & 21 \\ -2y + 3z & = & -13 \\ 4z & = & 12 \end{array}$$

Back-substitution tells us that the solution set is  $\{(-1, 11, 3)\}$ . So the solution set of the third system is not the same as the solution set of the first system. It means that they are not equivalent.

*Replacing systems by equivalent systems*

How do we know that simplifying a system of equations we obtain the one which is equivalent to the original system? It depends what we do to simplify the system. There are certain operations which for sure lead to equivalent system. Even more, they are general and can be used to solve any system of linear equations.

**First Operation: interchanging two equations.** Let us start with the simplest one: interchanging two equations. Changing the order of equations in the system does not effect the solutions set which is still the intersection of the same solution sets of individual equations.

5.5.3. EXAMPLE. Obviously, the following two systems have identical solution sets, so they are equivalent.

$$\begin{array}{rcl} 7x + 2y + 2z & = & 21 \\ - 2y + 3z & = & 1 \\ & & 4z = 12 \end{array}$$

$$\begin{array}{rcl} 7x + 2y + 2z & = & 21 \\ & & 4z = 12 \\ - 2y + 3z & = & 1 \end{array}$$

**Second Operation: multiplying an equation by a number.** The second operation which leads to the equivalent system is multiplying one equation in the original system by a number which not 0. We know that the solution set of the equation multiplied by a number does not change. So the solution set of the whole system does not change being the intersection of the same sets.

5.5.4. EXAMPLE. In the Example 5.5.2 we have checked that the two following equations have equal solution sets.

$$\begin{array}{rcl} 7x + 2y + 2z & = & 21 \\ - 2y + 3z & = & 1 \\ & & 4z = 12 \end{array}$$

$$\begin{array}{rcl} 21x + 6y + 6z & = & 63 \\ - 2y + 3z & = & 1 \\ & & 4z = 12 \end{array}$$

**Third Operation: adding a multiple of one equation to another equation.** The third operation which leads to the equivalent systems is the most useful because it changes a system significantly. It is adding a multiple of one equation in the system to another equation. Let us describe it closer. It is only between two equations in the system. The first equation is not changed but only used to change the second one. The second equation is changed in the following way: we replace it by the sum of the second and the first multiplied by a number. Let us illustrate it with an example.

5.5.5. EXAMPLE.

Let us consider the system

$$\begin{aligned} x + 2y &= 3 \\ 3x + 4y &= 2 \end{aligned}$$

We multiply the first equation by  $(-3)$  and add to the second equation. It gives us the new second equation.

$$\begin{aligned} x + 2y &= 3 \\ 3x + (-3)x + 4y + (-3)2y &= 2 + (-3)3 \end{aligned}$$

We obtain the system

$$\begin{aligned} x + 2y &= 3 \\ -2y &= -7 \end{aligned}$$

*The elimination method*

For two equivalent systems the one which has an equation with less number of variables is simpler to solve. Thus eliminating variables is one of the most powerful method of solving systems of equations. Each particular system equation that we are going to present could be solved in many different ways. Sometimes much simpler than what we present. However, our concern is to present the method which is not difficult and, at the same time, works in general for all linear systems. The simple form that we want to achieve is the form to which is possible to apply the back-substitution method. Let us start with the example.

### 5.5.6. EXAMPLE.

Let us consider the following system

$$\begin{aligned}3x + 4y &= 18 \\2x + 3y &= 13\end{aligned}$$

We want to eliminate the unknown  $x$  from the second equation. The  $x$ -coefficient in the first equation is 3 so we multiply the second equation by 3.

$$\begin{aligned}3x + 4y &= 18 \\6x + 9y &= 39\end{aligned}$$

Then we multiply the first equation multiply by  $(-2)$  and add to the second equation to obtain the new second equation.

$$\begin{array}{rccccccc}3x & & + & & 4y & & = & & 18 \\6x + (-2)3x & & + & & 9y + (-2)4y & & = & & 39 + (-2)18\end{array}$$

It gives

$$\begin{aligned}3x + 4y &= 18 \\y &= 3\end{aligned}$$

At this moment we are done with the elimination. To finish the problem up we use the back-substitution method which gives the solution set  $\{(2, 3)\}$ .

For the system in three variables the method of elimination is just longer but the steps are mainly the same.

### 5.5.7. EXAMPLE.

let us consider the following  $3 \times 3$  system

$$\begin{aligned}3x + y + 2z &= 13 \\2x + 3y + 4z &= 19 \\x + 4y + 3z &= 15\end{aligned}$$

We see that in the last equation the  $x$  coefficient is 1. It is very convenient because it would be an equation to use to eliminate  $x$  from the other equations. Let us move it to the front by changing the order of equations.

$$\begin{aligned}x + 4y + 3z &= 15 \\3x + y + 2z &= 13 \\2x + 3y + 4z &= 19\end{aligned}$$

Now the elimination starts. We add the first equation multiplied by  $(-3)$  to the second equation.

$$\begin{array}{rclcrcl} x & + & 4y & + & 3z & = & 15 \\ & & - & 11y & - & 7z & = & -32 . \\ 2x & + & 3y & + & 4z & = & 19 \end{array}$$

We add the first equation multiplied by  $(-2)$  to the third equation.

$$\begin{array}{rclcrcl} x & + & 4y & + & 3z & = & 15 \\ & & - & 11y & - & 7z & = & -32 . \\ & & - & 5y & - & 2z & = & -11 \end{array}$$

The system still does not fit to back-substitution. We need to eliminate the variable  $y$  from the last equation. In order to do it we need first to multiply the third equation by  $(-11)$ .

$$\begin{array}{rclcrcl} x & + & 4y & + & 3z & = & 15 \\ & & - & 11y & - & 7z & = & -32 . \\ & & & 55y & + & 22z & = & 121 \end{array}$$

Then we add the second equation multiplied by 5 to third equation.

$$\begin{array}{rclcrcl} x & + & 4y & + & 3z & = & 15 \\ & & - & 11y & - & 7z & = & -32 . \\ & & & & - & 13z & = & -39 \end{array}$$

Now we can use the back-substitution method. It gives us the solution set  $\{(2, 1, 3)\}$ .

### *Gaussian elimination for matrices*

When we perform the elimination there is a lot of writing. It is especially inconvenient to carry on the symbols of variables. The algebraic operations are done on the numbers only. So we can skip the symbols of variables and do the elimination on the augmented matrix associated to the system. It is called *Gaussian elimination* for matrices. We need to remember that not seeing a variable in the equation means the same as seeing 0 in the corresponding location in the augmented matrix. We illustrate Gaussian elimination for two variables first. To show the connection between the two methods we will use the augmented matrix of the system from the Example 5.5.6.

5.5.8. EXAMPLE.

$$\left( \begin{array}{cc|c} 3 & 4 & 18 \\ 2 & 3 & 13 \end{array} \right)$$

First we multiply the second row by the number 3

$$\left( \begin{array}{cc|c} 3 & 4 & 18 \\ (3) \cdot 2 & (3) \cdot 3 & (3) \cdot 13 \end{array} \right) = \left( \begin{array}{cc|c} 3 & 4 & 18 \\ 6 & 9 & 39 \end{array} \right).$$

Then we multiply the first row by the number  $(-2)$  and the result to the second to get the new second row

$$\left( \begin{array}{cc|c} 3 & 4 & 18 \\ 6 + (-2) \cdot 3 & 9 + (-2) \cdot 4 & 39 + (-2) \cdot 18 \end{array} \right) = \left( \begin{array}{cc|c} 3 & 4 & 18 \\ 0 & 1 & 3 \end{array} \right).$$

5.5.9. EXAMPLE.

$$\left( \begin{array}{ccc|c} 3 & 1 & 2 & 13 \\ 2 & 3 & 4 & 19 \\ 1 & 4 & 3 & 15 \end{array} \right)$$

First we interchange the rows

$$\left( \begin{array}{ccc|c} 1 & 4 & 3 & 15 \\ 3 & 1 & 2 & 13 \\ 2 & 3 & 4 & 19 \end{array} \right).$$

Then we multiply the first row by  $(-3)$  and add to the second row

$$\left( \begin{array}{ccc|c} 1 & 4 & 3 & 15 \\ 3 + (-3) \cdot 1 & 1 + (-3) \cdot 4 & 2 + (-3) \cdot 3 & 13 + (-3) \cdot 15 \\ 2 & 3 & 4 & 19 \end{array} \right) = \left( \begin{array}{ccc|c} 1 & 4 & 3 & 15 \\ 0 & -11 & -7 & -32 \\ 2 & 3 & 4 & 19 \end{array} \right).$$

Then we multiply the first row by  $(-2)$  and add to the third row

$$\left( \begin{array}{ccc|c} 1 & 4 & 3 & 15 \\ 0 & -11 & -7 & -32 \\ 2 + (-2) \cdot 1 & 3 + (-2) \cdot 4 & 4 + (-2) \cdot 3 & 19 + (-2) \cdot 15 \end{array} \right) = \left( \begin{array}{ccc|c} 1 & 4 & 3 & 15 \\ 0 & -11 & -7 & -32 \\ 0 & -5 & -2 & -11 \end{array} \right).$$

Then we multiply the third row by  $(-11)$

$$\left( \begin{array}{ccc|c} 1 & 4 & 3 & 15 \\ 0 & -11 & -7 & -32 \\ 0 & (-11) \cdot (-5) & (-11) \cdot (-2) & (-11) \cdot (-11) \end{array} \right) = \left( \begin{array}{ccc|c} 1 & 4 & 3 & 15 \\ 0 & -11 & -7 & -32 \\ 0 & 55 & 22 & 121 \end{array} \right).$$

Finally, we multiply the second row by 5 and add to the third row

$$\left( \begin{array}{ccc|c} 1 & 4 & 3 & 15 \\ 0 & -11 & -7 & -32 \\ 0 & 55 + 5 \cdot (-11) & 22 + 5 \cdot (-7) & 121 + 5 \cdot (-32) \end{array} \right) = \left( \begin{array}{ccc|c} 1 & 4 & 3 & 15 \\ 0 & -11 & -7 & -32 \\ 0 & 0 & -13 & -39 \end{array} \right).$$