5.6. Solving linear systems using determinants

Determinants of 2×2 matrices

5.6.1. DEFINITION. The determinant of a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is the number $ad - bc$. We write it

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

5.6.2. EXAMPLE.

To understand the above definition better let us describe the three steps to find the determinant of the matrix

$$\begin{pmatrix} 5 & 2 \\ 7 & 11 \end{pmatrix}.$$

Step 1. Find the product of the underlined entries on the first diagonal

$$\begin{pmatrix} \underline{5} & 2 \\ 7 & \underline{11} \end{pmatrix}$$

$$5 \cdot 11$$

Step 2. Find the product of the underlined entries on the other diagonal

$$\begin{pmatrix} 5 & \underline{2} \\ \underline{7} & 11 \end{pmatrix}$$

$$2 \cdot 7$$

Step 3. Subtract the products

$$5 \cdot 11 - 2 \cdot 7 = 55 - 14 = 41$$

Thus

$$\det \begin{pmatrix} 5 & 2 \\ 7 & 11 \end{pmatrix} = 41.$$

Cramer's Rule for systems in two variables

5.6.3. Cramer's Rule in two variables.

For the system of equations

$$\begin{array}{rcrcl} ax & + & by & = & l \\ cx & + & dy & = & m \end{array}$$

let

$$D = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad D_x = \det \begin{pmatrix} \underline{l} & b \\ \underline{m} & d \end{pmatrix}, \text{ and } \ D_y = \det \begin{pmatrix} a & \underline{l} \\ c & \underline{m} \end{pmatrix}.$$

If $D \neq 0$, then the solution to the system is given by

$$x = \frac{D_x}{D} \ \text{ and } \ y = \frac{D_y}{D}.$$

5.6.4. EXAMPLE.

Let us apply Cramer's Rule to solve the system

$$\begin{array}{rcrcl} 3x & - & 2y & = & 9 \\ -x & + & 3y & = & 3 \end{array}.$$

For this system the matrix of coefficients is

$$\begin{pmatrix} 3 & -2 \\ -1 & 3 \end{pmatrix}.$$

Thus the determinant D of the matrix of coefficients is

$$D = \det \begin{pmatrix} 3 & -2 \\ -1 & 3 \end{pmatrix} = 3 \cdot 3 - (-2) \cdot (-1) = 7.$$

In the matrix of coefficients $\begin{pmatrix} \underline{3} & -2 \\ \underline{-1} & 3 \end{pmatrix}$ the first column $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$

is called the x-column because it consists of the coefficients in the front of the variable x. In order to calculate the determinant D_x we replace the x-column by the column of constants

$$\begin{pmatrix} 9 \\ 3 \end{pmatrix}$$

which consists of the constant numbers which are on the right sides of the equations. It gives

$$\begin{pmatrix} \underline{3} & -2 \\ \underline{-1} & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} \underline{9} & -2 \\ \underline{3} & 3 \end{pmatrix}.$$

Then we calculate the determinant of the obtained matrix

$$D_x = \det \begin{pmatrix} \underline{9} & -2 \\ \underline{3} & 3 \end{pmatrix} = 9 \cdot 3 - (-2) \cdot 3 = 33.$$

The y-column of the matrix of coefficients $\begin{pmatrix} 3 & \underline{-2} \\ -1 & \underline{3} \end{pmatrix}$ is $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

We replace it by the column of constants $\begin{pmatrix} 9 \\ 3 \end{pmatrix}$.

It gives

$$\begin{pmatrix} 3 & \underline{-2} \\ -1 & \underline{3} \end{pmatrix} \Rightarrow \begin{pmatrix} 3 & \underline{9} \\ -1 & \underline{3} \end{pmatrix}.$$

Then we calculate the determinant

$$\det \begin{pmatrix} 3 & \underline{9} \\ -1 & \underline{3} \end{pmatrix} = 3 \cdot 3 - 9 \cdot (-1) = 18.$$

By Cramer's Rule

$$x = \frac{D_x}{D} = \frac{33}{7} \quad \text{and} \quad y = \frac{D_y}{D} = \frac{18}{7}.$$

Determinants of 3×3 *matrices*

5.6.5. DEFINITION.

For a 3×3 matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

the determinant is defined as

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} =$$

$$a_{11} \cdot a_{22} \cdot a_{33} + a_{12} \cdot a_{23} \cdot a_{31} + a_{21} \cdot a_{32} \cdot a_{13} - a_{13} \cdot a_{22} \cdot a_{31} - a_{12} \cdot a_{21} \cdot a_{33} - a_{11} \cdot a_{23} \cdot a_{32}.$$

In the above definition the determinant of an 3×3 matrix is the sum of six terms. Three of the terms have $+$ in the front and the three other $-$. Each term consists of the product of three entries from the given matrix. In the example below we give a convenient way of writing of the six terms which define the determinant.

5.6.6. EXAMPLE. We find the determinant of the matrix

$$\begin{pmatrix} -2 & -3 & -1 \\ -7 & 4 & 5 \\ 0 & 6 & 1 \end{pmatrix}.$$

We start with repeating twice the given matrix

$$\begin{pmatrix} -2 & -3 & -1 & \underline{-2} & \underline{-3} & \underline{-1} \\ -7 & 4 & 5 & \underline{-7} & \underline{4} & \underline{5} \\ 0 & 6 & 1 & \underline{0} & \underline{6} & \underline{1} \end{pmatrix}.$$

The first diagonal of the matrix is

$$\begin{pmatrix} \underline{-2} & -3 & -1 \\ -7 & \underline{4} & 5 \\ 0 & 6 & \underline{1} \end{pmatrix}.$$

It has the entries $(-2, 4, 1)$ Then we form three lines parallel to the first diagonal of the given matrix

$$\begin{pmatrix} \underline{-2} & \underline{-3} & \underline{-1} & -2 & -3 & -1 \\ -7 & \underline{4} & \underline{5} & \underline{-7} & 4 & 5 \\ 0 & 6 & \underline{1} & \underline{0} & \underline{6} & 1 \end{pmatrix}.$$

They have the entries $(-2, 4, 1)$, $(-3, -7, 0)$, and $(-1, -7, 6)$, respectively. We form the first three products with the sign $+$ in the front

$$+(-2) \cdot 4 \cdot 1 + (-3) \cdot (-7) \cdot 0 + (-1) \cdot (-7) \cdot 6 = 34$$

Now we repeat the same process for the other diagonal of the given matrix

$$\begin{pmatrix} -2 & -3 & \underline{-1} \\ -7 & \underline{4} & 5 \\ \underline{0} & 6 & 1 \end{pmatrix}$$

which has the entries $(-1, 4, 0)$. Then we form three lines parallel to the second diagonal of the given matrix

$$\begin{pmatrix} -2 & -3 & \underline{-1} & \underline{-2} & \underline{-3} & -1 \\ -7 & \underline{4} & \underline{5} & \underline{-7} & 4 & 5 \\ \underline{0} & \underline{6} & \underline{1} & 0 & 6 & 1 \end{pmatrix}.$$

They have entries $(-1, 4, 0)$;, $(-2, 5, 6)$, $(-3, -7, 1)$, respectively. We from the other three products with the sign $-$ in the front

$$-(-1) \cdot 4 \cdot 0 - (-2) \cdot 5 \cdot 6 - (-3) \cdot (-7) \cdot 1 = 39$$

Finally, we find that

$$\det \begin{pmatrix} -2 & -3 & -1 \\ -7 & 4 & 5 \\ 0 & 6 & 1 \end{pmatrix} = 34 + 39 = 73.$$

5.6.7. Cramer's Rule for three variables.
For the system of equations

$$\begin{array}{ccccccc} a_1x & + & b_1y & + & c_1z & = & L \\ a_2x & + & b_2y & + & c_2z & = & K \\ a_3x & + & b_3y & + & c_3z & = & M \end{array}$$

let

$$D = \det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \;, \quad D_x = \det \begin{pmatrix} L & b_1 & c_1 \\ K & b_2 & c_2 \\ M & b_3 & c_3 \end{pmatrix},$$

$$D_y = \det \begin{pmatrix} a_1 & L & c_1 \\ a_2 & K & c_2 \\ a_3 & M & c_3 \end{pmatrix} \text{, and } \; D_z = \det \begin{pmatrix} a_1 & b_1 & L \\ a_2 & b_2 & K \\ a_3 & b_3 & M \end{pmatrix}.$$

If $D \neq 0$, then the solution to the system is given by

$$x = \frac{D_x}{D} \;, \quad y = \frac{D_y}{D} \text{ , and } \; z = \frac{D_z}{D}.$$

5.6.8. EXAMPLE.
We will solve the system below using Cramer's Rule.

$$\begin{array}{ccccccc} x & + & y & + & z & = & 0 \\ 2x & - & y & + & z & = & -1 \\ -x & + & 3z & - & z & = & -8 \end{array}$$

The matrix of coefficients is $\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ -1 & 3 & -1 \end{pmatrix}$ and the column of constants is $\begin{pmatrix} 0 \\ -1 \\ -8 \end{pmatrix}$.

We will evaluate the determinants D, D_x, D_y, D_z.

We start with the determinant of the matrix of coefficients

$$D = \det \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ -1 & 3 & -1 \end{pmatrix} = 4.$$

Then we replace the x-column in the matrix of coefficients by the column of constants

$$\begin{pmatrix} \underline{1} & 1 & 1 \\ \underline{2} & -1 & 1 \\ \underline{-1} & 3 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} \underline{0} & 1 & 1 \\ \underline{-1} & -1 & 1 \\ \underline{-8} & 3 & -1 \end{pmatrix}$$

and calculate the determinant of the resulting matrix

$$D_x = \det \begin{pmatrix} \underline{0} & 1 & 1 \\ \underline{-1} & -1 & 1 \\ \underline{-8} & 3 & -1 \end{pmatrix} = -20.$$

We replace the y-column in the matrix of coefficients by the column of constants

$$\begin{pmatrix} 1 & \underline{1} & 1 \\ 2 & \underline{-1} & 1 \\ -1 & \underline{3} & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & \underline{0} & 1 \\ 2 & \underline{-1} & 1 \\ -1 & \underline{-8} & -1 \end{pmatrix}$$

and calculate the determinant of the resulting matrix

$$D_y = \det \begin{pmatrix} 1 & \underline{0} & 1 \\ 2 & \underline{-1} & 1 \\ -1 & \underline{-8} & -1 \end{pmatrix} = -8.$$

Finally we replace the z-column in the matrix of coefficients by the column of constants

$$\begin{pmatrix} 1 & 1 & \underline{1} \\ 2 & -1 & \underline{1} \\ -1 & 3 & \underline{-1} \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & \underline{0} \\ 2 & -1 & \underline{-1} \\ -1 & 3 & \underline{-8} \end{pmatrix}$$

and calculate the determinant of the resulting matrix

$$D_z = \det \begin{pmatrix} 1 & 1 & \underline{0} \\ 2 & -1 & \underline{-1} \\ -1 & 3 & \underline{-8} \end{pmatrix} = 28$$

Now, by Cramer's Rule

$$x = \frac{D_x}{D} = \frac{-20}{4} = -5, \;\; y = \frac{D_y}{D} = \frac{-8}{4} = -2, \;\; z = \frac{D_z}{D} = \frac{28}{4} = 7.$$

The solution set to the system is $\{(-5, -2, 7)\}$.