6.1. Sequences

Sequences as functions

By $\mathbf{N}$ we denote the set of all natural (counting) numbers i.e.

$$\mathbf{N} = \{1, 2, 3, \ldots\}.$$

If $y = f(x)$ is a function with the domain $(0, \infty)$, then the values $f(1), f(2), f(3), \ldots$ form a sequence.

6.1.1. EXAMPLE. For instance, let $f(x) = x^2 + 3$. Then

$$f(1) = 1^2 + 3 = 4, f(2) = 2^2 + 3 = 7, f(3) = 3^2 + 3 = 12, \ldots$$

form a sequence. The first term of this sequence is 4, the second is 7, the third is 12 and so on. We can tell each term of the sequence. For instance the eleventh term is $11^2 + 3 = 124$ and the twenties element is $20^2 + 3 = 403$. In general, the term which is on the place numbered by n (i.e. nth term) is $n^2 + 3$.

6.1.2. DEFINITION. A sequence is a function whose domain is the set of all natural numbers $\{1, 2, 3, \ldots\}$.

Informally, we can think of a sequence of numbers as an ordered list of numbers.

Sequential notation

For the independent variable of a sequence, we usually use n. For the dependent variable $f(n)$, we generally write a_n and we read it "a sub n". So the sequence from the Example 6.1.1 is also defined by

$$a_n = n^2 + 3.$$

6.1.3. EXAMPLE. The sequence of positive even integers $2, 4, 6, \ldots$ can be defined as

$$a_n = 2n.$$

6.1.4. EXAMPLE. The sequence of positive odd integers $1, 3, 5, \ldots$ is defined by

$$a_= 2n - 1.$$

6.1.5. EXAMPLE. The sequence $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \ldots$ is defined by

$$a_n = \frac{n+1}{n}.$$

6.1.6. EXAMPLE. What is the fifth term of the sequence $1, 8, 27, 64, \ldots$?

We need to determine the pattern between the terms.
First term is 1 which is the same as 1^3.
Second term is 8 which is the same as 2^3.
Third term is 27 which is the same as 3^3.
Fourth term is 64 which is the same as 4^3.
Hence, the fifth term is 125 because $5^3 = 125$.

6.1.7. EXAMPLE. What is the nth term of the sequence $0, 3, 8, 15, 24, \ldots$? For convenience, we will use the sequential notation. We have

$$\begin{aligned} a_1 &= 0 = 1^2 - 1 \\ a_2 &= 3 = 2^2 - 1 \\ a_3 &= 8 = 3^2 - 1 \\ a_4 &= 15 = 4^2 - 1 \\ a_5 &= 24 = 5^2 - 1. \end{aligned}$$

Hence the expression for the nth term is

$$a_n = n^2 - 1.$$

Factorial notation

We need a notation for a product of consecutive integers because it occurs very often when we talk about sequences.

6.1.8. DEFINITION. For any positive integer n, the notation $n!$ (read as "n factorial") is defined by

$$n! = n \cdot (n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1.$$

In addition we define

$$0! = 1.$$

From the definition we have

$$n \cdot ((n-1)!) = n!.$$

If $n = 1$, then $1 \cdot (0!) = 1! = 1$. Hence $0! = 1$.

6.1.9. EXAMPLE.

$$2! = 2 \cdot 1 = 2$$

$$3! = 3 \cdot 2 \cdot 1 = 6$$

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120.$$

6.1.10. EXAMPLE. What is the fifth term of the sequence

$$a_n = \frac{n!}{(n-1)!}?$$

If $n = 5$ then

$$a_5 = \frac{5!}{4!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{5}{1} = 5.$$