6.3. Geometric sequence.

A sequence in which each term after the first is a constant multiple of the preceding term is called a geometric sequence.

Definition of a Geometric Sequence:

> A sequence is geometric if each term after the first is obtained by multiplying by a nonzero fixed number (positive or negative) to the preceding term. The sequence, $a_1, a_2, a_3, \ldots, a_n$, is geometric if there is a number r such that $r = a_2 \div a_1$, $a_3 \div a_2$, and so on. The number r is called the common ratio.

Example: The sequence, 2, 6, 18, is geometric since the ratio between two adjacent terms is always 3. That is, each term multiplied by 3 will yield the next term.

Example: The first five terms of a geometric sequence with a first term of 3 and a common ratio –2 can be found as follows.

$a_1 = 3$

$a_2 = 3 \cdot (-2) = -6$

$a_3 = (-6) \cdot (-2) = 12$

$a_4 = 12 \cdot (-2) = -24$

$a_5 = (-24) \cdot (-2) = 48$

Hence, the first five terms are: 3, –6, 12, –24, and 48.

Example: Find r for the sequence, 15, 15/3, 15/9, 15/27,….

$r = a_2 \div a_1$

$r = (15/3) \div 15$

$r = 1/3.$

The **general term of a geometric sequence (a_n)** with a first term of a_1 and a common ratio of r is $a_n = a_1(r^{n-1})$.

Example: The general term of the geometric sequence with a first term of –2 and a common ratio of 3 is:

$a_n = a_1(r^{n-1})$

$a_n = -2(3)^{n-1}$

Example: To find the general term of the geometric sequence, –4, 8, begin by finding r, the common ratio.

$r = a_2 \div a_1$

$r = 8 \div -4$

$r = -2.$

Once r is found, use the formula for the general term.

$a_n = a_1(rn-1)$

$a_n = -4(-2)n-1$

Example: To find the eleventh term (a_{11}) of the geometric sequence with a first term of 3 and a common ratio of –4, one uses the general form:

$a_n = a_1(rn-1)$

$a_{11} = 3(-4)11-1$

$a_{11} = 3(-4)10$

$a_{11} = 3(1{,}048{,}576)$

$a_{11} = 3{,}145{,}728.$

Example: To find the seventh term (a_7) of the geometric sequence with a first term of 6 and a third term of 54, one uses the general form to first find r.

$a_n = a_1(rn-1)$

$a_3 = 6(r)3-1$

$54 = 6r2$

$r2 = 9$

$r = \pm 3.$

Then one uses the general form to find a_7.

$a_7 = 6(\pm 3)7\text{-}1$

$a_7 = 6(\pm 3)6$

$a_7 = 6(729)$

$a_7 = 4374$.

Example: If $a_5 = 48$ and $a_8 = –384$, then to find a_{10} one must use the general form of a geometric sequence three times.

$a_5 = a_1(r5\text{-}1)$

$48 = a_1(r4)$

<u>or</u>

$a_1(r4) = 48$ (Equation 1).

$a_8 = a_1(r8\text{-}1)$

$–384 = a_1(r7)$

<u>or</u>

$a_1(r7) = –384$ (Equation 2).

Now solving Equation 1 and Equation 2 by applying what was learned in module 5 (substitution method), one determines that $r = 2$ and $a_1 = 3$.

Now one has the necessary information to calculate a_{10}.

$a_{10} = a_1(r10\text{-}1)$

$a_{10} = 3(2)9$

$a_{10} = 3(512)$

$a_{10} = 1536$.

Example: To find the number of terms in the finite sequence, 4, 2,,1/16, one must use the general form with $a_n = 1/16$, $a_1 = 4$, and $r = 2 \div 4 = 1/2$.

1/16 = 4(1/2)n-1

1/64 = (1/2)n-1

1/64 = (1/2)n · (1/2)-1

1/128 = (1/2)n

n = 7.

Thus, there are a total of 7 terms in the given geometric sequence.

Note: *The form for the general term of a geometric sequence can be very useful.*

To find the **sum of the first *n* terms of a geometric sequence** with first term a_1, and common ratio *r*, one may use the following formula:

$$s_n = \frac{a_1\left(r^n - 1\right)}{r - 1} \quad (r \neq 1).$$

Example: Find the sum of the first six terms of the geometric sequence with first term –3and common ratio 4.

Note: *Substitute n = 6, a_1 = –3, and r = 4 into the formula for sum of the first n terms of a geometric sequence.*

$$s_6 = \frac{-3\left(4^6 - 1\right)}{6 - 1}$$

$$s_6 = \frac{-3(4095)}{5}$$

$$s_6 = -2457$$

Example: Find the sum of the first five terms of the geometric sequence, 1/3, 1/9, 1/27,.

Note: *Begin by finding r by using the fact that r = $a_2 \div a_1$. Then use the formula for sum with a_1 = 1/3, and n = 5.*

r = 1/9 ÷ 1/3 = 1/3.

$$s_5 = \frac{\frac{1}{3}\left(\left(\frac{1}{3}\right)^5 - 1\right)}{5-1}$$

$$s_5 = \frac{\left(\frac{1}{3}\right)^6 - \frac{1}{3}}{4}$$

$$s_5 = -0.082990398..$$

Example: Evaluate:

$$\sum_{i=1}^{7} 4\left(\frac{2}{5}\right)^i.$$

Note: *Begin by finding a_1, and r. Then use the formula for sum with n = 7.*

$a_1 = 4(2/5)^1 = 8/5$.

$a_2 = 4(2/5)^2 = 16/25$.

$r = a_2 \div a_1 = 16/25 \div 8/5 = 2/5$.

$$s_7 = \frac{\frac{8}{5}\left(\left(\frac{2}{5}\right)^7 - 1\right)}{7-1}$$

$$s_7 = -0.26622976\ldots$$

To find the **sum of an infinite geometric sequence** with first term a_1, and common ratio *r*, where $0 < r < 1$ use the following formula:

$$s_\infty = \frac{a_1}{1-r}.$$

If $r > 1$ or $r < -1$, then the sum does not exist.

Example: Find the sum of the infinite geometric sequence with $a_1 = 6$ and $r = 1/3$.

$$s_\infty = \frac{6}{1-\frac{1}{3}}$$

$$s_\infty = 9.$$

Example: Evaluate:

$$\sum_{i=1}^{\infty} \frac{3}{5}\left(\frac{5}{6}\right)^i.$$

Note: *Begin by finding a_1, and r. Then use the formula for the sum of an infinite geometric sequence.*

$$a_1 = \frac{3}{5}\left(\frac{5}{6}\right)^1 = \frac{1}{2}.$$

$$a_2 = \frac{3}{5}\left(\frac{5}{6}\right)^2 = \frac{5}{12}$$

$$r = a_2 \div a_1 = \frac{5}{12} \div \frac{1}{2} = \frac{5}{6}.$$

$$s_\infty = \frac{\frac{1}{2}}{1-\frac{5}{6}} = 3.$$

The formula for the sum of an infinite geometric sequence can also be used to change a repeating decimal to a rational number.

Example: Change 1.414141… to a rational number.

Note: *A number a.bcde… can be written as:*

$$a + \frac{b}{10} + \frac{c}{100} + \frac{d}{1000} + \ldots.$$

Hence, $1.414141\ldots = 1 + \frac{41}{100} + \frac{41}{10000} + \frac{41}{1000000} + \ldots$**.**

Since $a_1 = \frac{41}{100}$ **and** $r = \frac{41}{10000} \div \frac{41}{100} = \frac{1}{100}$ **then**

$$s_\infty = 1 + \frac{\frac{41}{100}}{1 - \frac{1}{100}}$$

$$s_\infty = 1 + \frac{41}{99}$$

$$s_\infty = \frac{140}{99}.$$