

6.3. Geometric sequence.

A sequence in which each term after the first is a constant multiple of the preceding term is called a **geometric sequence**.

Definition of a Geometric Sequence:

A sequence is geometric if each term after the first is obtained by multiplying by a nonzero fixed number (positive or negative) to the preceding term. The sequence, $a_1, a_2, a_3, \dots, a_n$, is geometric if there is a number r such that $r = a_2 \div a_1, a_3 \div a_2$, and so on. The number r is called the **common ratio**.

Example: The sequence, 2, 6, 18, is geometric since the ratio between two adjacent terms is always 3. That is, each term multiplied by 3 will yield the next term.

Example: The first five terms of a geometric sequence with a first term of 3 and a common ratio -2 can be found as follows.

$$a_1 = 3$$

$$a_2 = 3 \cdot (-2) = -6$$

$$a_3 = (-6) \cdot (-2) = 12$$

$$a_4 = 12 \cdot (-2) = -24$$

$$a_5 = (-24) \cdot (-2) = 48$$

Hence, the first five terms are: 3, -6 , 12, -24 , and 48.

Example: Find r for the sequence, 15, $15/3$, $15/9$, $15/27, \dots$

$$r = a_2 \div a_1$$

$$r = (15/3) \div 15$$

$$r = 1/3.$$

The **general term of a geometric sequence (a_n)** with a first term of a_1 and a common ratio of r is $a_n = a_1(r^{n-1})$.

Example: The general term of the geometric sequence with a first term of -2 and a common ratio of 3 is:

$$a_n = a_1(r^{n-1})$$

$$a_n = -2(3)^{n-1}$$

Example: To find the general term of the geometric sequence, $-4, 8$, begin by finding r , the common ratio.

$$r = \frac{a_2}{a_1}$$

$$r = 8 \div -4$$

$$r = -2.$$

Once r is found, use the formula for the general term.

$$a_n = a_1(r^{n-1})$$

$$a_n = -4(-2)^{n-1}$$

Example: To find the eleventh term (a_{11}) of the geometric sequence with a first term of 3 and a common ratio of -4 , one uses the general form:

$$a_n = a_1(r^{n-1})$$

$$a_{11} = 3(-4)^{11-1}$$

$$a_{11} = 3(-4)^{10}$$

$$a_{11} = 3(1,048,576)$$

$$a_{11} = 3,145,728.$$

Example: To find the seventh term (a_7) of the geometric sequence with a first term of 6 and a third term of 54, one uses the general form to first find r .

$$a_n = a_1(r^{n-1})$$

$$a_3 = 6(r)^{3-1}$$

$$54 = 6r^2$$

$$r^2 = 9$$

$$r = \pm 3.$$

Then one uses the general form to find a_7 .

$$a_7 = 6(\pm 3)^{7-1}$$

$$a_7 = 6(\pm 3)^6$$

$$a_7 = 6(729)$$

$$a_7 = 4374.$$

Example: If $a_5 = 48$ and $a_8 = -384$, then to find a_{10} one must use the general form of a geometric sequence three times.

$$a_5 = a_1 (r^{5-1})$$

$$48 = a_1 (r^4)$$

or

$$a_1 (r^4) = 48 \text{ (Equation 1).}$$

$$a_8 = a_1 (r^{8-1})$$

$$-384 = a_1 (r^7)$$

or

$$a_1 (r^7) = -384 \text{ (Equation 2).}$$

Now solving **Equation 1** and **Equation 2** by applying what was learned in module 5 (**substitution method**), one determines that $r = 2$ and $a_1 = 3$.

Now one has the necessary information to calculate a_{10} .

$$a_{10} = a_1 (r^{10-1})$$

$$a_{10} = 3(2)^9$$

$$a_{10} = 3(512)$$

$$a_{10} = 1536.$$

Example: To find the number of terms in the finite sequence, 4, 2, 1/16, one must use the general form with $a_n = 1/16$, $a_1 = 4$, and $r = 2 \div 4 = 1/2$.

$$1/16 = 4(1/2)^{n-1}$$

$$1/64 = (1/2)^{n-1}$$

$$1/64 = (1/2)^n \cdot (1/2)^{-1}$$

$$1/128 = (1/2)^n$$

$$n = 7.$$

Thus, there are a total of 7 terms in the given geometric sequence.

Note: The form for the general term of a geometric sequence can be very useful.

To find the **sum of the first n terms of a geometric sequence** with first term a_1 , and common ratio r , one may use the following formula:

$$S_n = \frac{a_1(r^n - 1)}{r - 1} \quad (r \neq 1).$$

Example: Find the sum of the first six terms of the geometric sequence with first term -3 and common ratio 4 .

Note: Substitute $n = 6$, $a_1 = -3$, and $r = 4$ into the formula for sum of the first n terms of a geometric sequence.

$$S_6 = \frac{-3(4^6 - 1)}{4 - 1}$$

$$S_6 = \frac{-3(4095)}{3}$$

$$S_6 = -2457$$

Example: Find the sum of the first five terms of the geometric sequence, $1/3, 1/9, 1/27, \dots$.

Note: Begin by finding r by using the fact that $r = a_2 \div a_1$. Then use the formula for sum with $a_1 = 1/3$, and $n = 5$.

$$r = 1/9 \div 1/3 = 1/3.$$

$$s_5 = \frac{\frac{1}{3} \left(\left(\frac{1}{3} \right)^5 - 1 \right)}{5 - 1}$$

$$s_5 = \frac{\left(\frac{1}{3} \right)^6 - \frac{1}{3}}{4}$$

$$s_5 = -0.082990398\dots$$

Example: Evaluate:

$$\sum_{i=1}^7 4 \left(\frac{2}{5} \right)^i.$$

Note: Begin by finding a_1 , and r . Then use the formula for sum with $n = 7$.

$$a_1 = 4 \left(\frac{2}{5} \right)^1 = 8/5.$$

$$a_2 = 4 \left(\frac{2}{5} \right)^2 = 16/25.$$

$$r = a_2 \div a_1 = 16/25 \div 8/5 = 2/5.$$

$$s_7 = \frac{\frac{8}{5} \left(\left(\frac{2}{5} \right)^7 - 1 \right)}{7 - 1}$$

$$s_7 = -0.26622976\dots$$

To find the **sum of an infinite geometric sequence** with first term a_1 , and common ratio r , where $0 < r < 1$ use the following formula:

$$s_{\infty} = \frac{a_1}{1 - r}.$$

If $r > 1$ or $r < -1$, then the sum does not exist.

Example: Find the sum of the infinite geometric sequence with $a_1 = 6$ and $r = 1/3$.

$$s_{\infty} = \frac{6}{1 - \frac{1}{3}}$$

$$s_{\infty} = 9.$$

Example: Evaluate:

$$\sum_{i=1}^{\infty} \frac{3}{5} \left(\frac{5}{6}\right)^i.$$

Note: Begin by finding a_1 , and r . Then use the formula for the sum of an infinite geometric sequence.

$$a_1 = \frac{3}{5} \left(\frac{5}{6}\right)^1 = \frac{1}{2}.$$

$$a_2 = \frac{3}{5} \left(\frac{5}{6}\right)^2 = \frac{5}{12}$$

$$r = a_2 \div a_1 = \frac{5}{12} \div \frac{1}{2} = \frac{5}{6}.$$

$$s_{\infty} = \frac{\frac{1}{2}}{1 - \frac{5}{6}} = 3.$$

The formula for the sum of an infinite geometric sequence can also be used to change a repeating decimal to a rational number.

Example: Change 1.414141... to a rational number.

Note: A number $a.bcde\dots$ can be written as:

$$a + \frac{b}{10} + \frac{c}{100} + \frac{d}{1000} + \dots$$

Hence, $1.414141\dots = 1 + \frac{41}{100} + \frac{41}{10000} + \frac{41}{1000000} + \dots$

Since $a_1 = \frac{41}{100}$ **and** $r = \frac{41}{10000} \div \frac{41}{100} = \frac{1}{100}$ **then**

$$s_{\infty} = 1 + \frac{\frac{41}{100}}{1 - \frac{1}{100}}$$

$$s_{\infty} = 1 + \frac{41}{99}$$

$$s_{\infty} = \frac{140}{99}.$$