6.5. The Binomial Theorem

Recall that a binomial is a polynomial that has two terms which are either added or subtracted. In this lesson, you will study a formula that aids in raising a binomial to a specified power. The expansion of $(x + y)^n$ for several values of n is illustrated below.

$(x + y)^0 = 1$

$(x + y)^1 = x + y$

$(x + y)^2 = x^2 + 2xy + y^2$

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

There are several observations you can make about these expansions.

1. In each expansion, there are *n + 1* (original exponent plus one) terms.
2. In each expansion, *x* and *y* have symmetrical roles. That is, the powers of *x* decrease by 1 in successive terms while the powers of *y* increase by 1.
3. The first term is x^n and the last term is y^n.
4. The sum of the powers of each term is *n* (original exponent).
5. The coefficients increase and decrease in a symmetrical pattern.

The coefficients of a binomial expansion are called binomial coefficients and can be found by using the **binomial theorem**.

The Binomial Theorem: For any positive integer *n*:

$$(x+y)^n = x^n + \frac{n!}{(n-1)!\,1!}x^{n-1}y + \frac{n!}{(n-2)!\,2!}x^{n-2}y^2 + \frac{n!}{(n-3)!\,3!}x^{n-3}y^3 + \ldots + \frac{n!}{1!\,(n-1)!}xy^{n-1} + y^n.$$

Note: *n-factorial (n!) was discussed in lesson 1 of this module.*

Example: Expand $(2m + 3)^4$ by using the binomial theorem.

$$\begin{aligned}(2m+3)^4 &= (2m)^4 + \frac{4!}{3!\,1!}(2m)^3(3) + \frac{4!}{2!\,2!}(2m)^2(3)^2 + \frac{4!}{1!\,3!}(2m)(3)^3 + (3)^4 \\ &= 16m^4 + 4(8m^3)(3) + 6(4m^2)(9) + 4(2m)(27) + 81 \\ &= 16m^4 + 96m^3 + 216m^2 + 216m + 81.\end{aligned}$$

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Example: Expand $(a - 2b)^5$ using the binomial theorem.

Note: *Rewrite* $(a - 2b)^5$ *as* $(a + -2b)^5$.

$$\begin{aligned}(a + -2b)^5 &= a^5 + \frac{5!}{4!\,1!}a^4(-2b) + \frac{5!}{3!\,2!}a^3(-2b)^2 + \frac{5!}{2!\,3!}a^2(-2b)^3 + \frac{5!}{1!\,4!}a(-2b)^4 + (-2b)^5 \\ &= a^5 + 5a^4(-2b) + 10a^3(4b^2) + 10a^2(-8b^3) + 5a(16b^4) + (-32b^5) \\ &= a^5 - 10a^4b + 40a^3b^2 - 80a^2b^3 + 80ab^4 - 32b^5.\end{aligned}$$

The binomial theorem can also be written in summation notation as:

$$\sum_{i=0}^{n} \frac{n!}{(n-i)!\,i!} x^{n-i} y^i$$

Note: *The letter i is used because of the use of summation notion. It is not the imaginary number i.*

There is an easy way to remember the pattern for binomial coefficients. Arranging the coefficients in triangular form yields what is called **Pascal's Triangle**.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
and so on

Note: *The first and the last numbers in each row of Pascal's triangle are always 1. Every other number in each row is found by adding the two numbers immediately above the number. Hence, this pattern continues forever.*

Notice how the coefficients in each of the binomial expansions appear in Pascal's Triangle.

$(x + y)^0 = 1$
$(x + y)^1 = 1x + 1y$
$(x + y)^2 = 1x^2 + 2xy + 1y^2$
$(x + y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$

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$$(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$
$$(x + y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$
and so on

Example: Expand $(2a - 3)^4$ using Pascal's Triangle.

Note: *Since the initial exponent is 4, there are 4 + 1 = 5 terms in the expansion. Hence, use the coefficients from the fifth row of Pascal's Triangle.*

$$(2a - 3)^4 = (2a + -3)^4$$
$$= \mathbf{1}(2a)^4 + \mathbf{4}(2a)^3(-3) + \mathbf{6}(2a)^2(-3)^2 + \mathbf{4}(2a)(-3)^3 + \mathbf{1}(-3)^4$$
$$= 16a^4 - 96a^3 + 216a^2 - 216a + 81.$$

Any single term of a binomial expansion can be found without writing out the whole expansion. For instance, the 10th term of $(x + y)^n$ has y raised to the 9th power (since y has a power of 1 in the second term, a power of 2 in the third term, and so on). Since the exponents on x and y in any term must have a sum of n, the exponent on x in the 10th term is $n - 9$. These values, 9 and $n - 9$, determine the factorials in the denominator of the coefficient. Thus, the 10th term of $(x + y)^n$ is:

$$\frac{n!}{(n-9)!\,9!} x^{n-9} y^9.$$

Formula for the rth Term of the Binomial Expansion

If $n \geq r - 1$, then the rth term of the binomial expansion of $(x + y)^n$ is:

$$\frac{n!}{(n-r+1)!\,(r-1)!} x^{n-r+1} y^{r-1}.$$

Example: Find the 4th term of $(a + 2b)^{10}$.

Note: *Use the formula for the rth term of the binomial expansion and remember to start with the exponent on y which is 1 less than the term number r. Then subtract that exponent from n to get the exponent on x. These two exponents are then used as factorials in the denominator of the coefficient.*

In the 4th term, **2b** has an exponent of **4 – 1 = 3** and **a** has an exponent of **10 – 3 = 7**. Thus, the 4th term is:

$$\frac{10!}{7!\,3!}(a)^7(2b)^3 = \frac{10 \bullet 9 \bullet 8}{3 \bullet 2 \bullet 1}a^7\left(8b^3\right)$$
$$= 120a^7\left(8b^3\right)$$
$$= 960a^7b^3.$$

Example: Find the 5th term of $(a - 3b)^{12}$.

In the 5^{th} term, $-3b$ has an exponent of **5 – 1 = 4** and **a** has an exponent of **12 – 4 =8**. Thus, the 5th term is:

$$\frac{12!}{8!\,4!}(a)^8(-3b)^4 = \frac{12 \bullet 11 \bullet 10 \bullet 9}{4 \bullet 3 \bullet 2 \bullet 1}a^8\left(81b^4\right)$$
$$= 495\,a^8\left(81\,b^4\right)$$
$$= 40{,}095\,a^8b^4.$$