

6.6. Combinations and permutations.

Simple counting problems allow one to list each possible way that an event can occur. However, some events can occur in so many different ways that it would be difficult to write out an entire list. Hence, one must use the **fundamental counting principle**.

Fundamental Counting Principle:

If one event can occur in m ways and a second event can occur in n ways, then both events can occur in $m \cdot n$ ways, provided the outcome of the first event does not influence the outcome of the second event.

Note: *The fundamental principle of counting can be extended to any number of events. When the outcome of one event does not influence the outcome of the other, such events are said to be **independent events**.*

Example: A restaurant offers a choice of 2 salads, 6 main dishes, 4 side dishes, and 3 desserts. How many different 4-course meals can be selected?

Note: *Four independent events are involved: selecting a salad, selecting a main dish, selecting a side dish, and selecting a dessert.*

The first event can occur in 2 ways, the second in 6 ways, the third in 4 ways, and the fourth in 3 ways. Thus, there are

$$2 \cdot 6 \cdot 4 \cdot 3 = 144 \text{ possible meals.}$$

Example: David has 6 books that he wishes to arrange on his desk. How many different arrangements are possible?

Note: *Six events are involved: selecting a book for the first spot, selecting a book for the second spot, and so on. The outcome of the first event does not influence the outcome of the other events since one book has already been chosen.*

David has 6 choices for the first spot, 5 choices for the second spot, 4 choices for the third spot, and so on. Thus, there are

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720 \text{ different arrangements for the books.}$$

Example: How many different 7-digit telephone numbers are possible if the first digit can not be zero and no digit may repeat?

Note: *Our number system consists of 10 digits, which are $\{0, 1, 2, 3, 4, 5, 6, 7, 8, \text{ and } 9\}$.*

Since the first digit can not be a zero, then there are 9 choices for the first digit, 9 choices for the second digit since a zero can be used and no digits may repeat, 8 choices for the third digit, 7 choices for the fourth digit, and so on. Thus, there are

$9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 544,320$ possible telephone numbers.

The number of ways that n elements can be arranged in order is called a **permutation** of the elements.

Permutations of n Elements r at a Time

If $P(n, r)$ denotes the number of permutations of n elements taken r at a time with $r \leq n$, then

$$P(n, r) = \frac{n!}{(n-r)!}.$$

Some other symbols used for permutations of n things taken r at a time are ${}_n P_r$.

Clue Words: Arrangement, Schedule, Order

Example: Suppose that 7 people enter a swim meet. Assuming that there are no ties, in how many ways could the gold, silver, and bronze medals be awarded?

Note: Since awards are to be made, order is important. Thus, this would be treated as a permutation.

$$\begin{aligned} P(7, 3) &= \frac{7!}{(7-3)!} \\ &= \frac{7!}{4!} \\ &= \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} \\ &= 7 \cdot 6 \cdot 5 \\ &= 210. \end{aligned}$$

There are 210 ways that the medals could be awarded.

Example: In how many ways can 6 students be seated in a row of 6 desks?

Note: This is a permutation because order is important. That is, a student may wish to sit in a particular seat or behind/in front of a particular student.

$$\begin{aligned}P(6, 6) &= \frac{6!}{(6-6)!} \\ &= \frac{6!}{0!} \\ &= \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1} \\ &= 720.\end{aligned}$$

There are 720 ways the students could be seated.

Note: Most calculators have a key for permutations.

When the arrangement or order of the elements is not important this is called a **combination** of the elements.

Combinations of n Elements r at a Time

If $\binom{n}{r}$ represents the number of combinations of n elements taken r at a time with $r \leq n$, then

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

Other symbols used for combinations of n things taken r at a time are $C(n, r)$, ${}_nC_r$.

Clue Words: Group, Committee, Sample

Example: How many different committees of 3 people can be chosen to work on a special project from a group of 7 people?

Note: Since the order in which the members of the committee are chosen does not affect the result, this is a combination.

$$\begin{aligned}
\binom{7}{3} &= \frac{7!}{(7-3)!3!} \\
&= \frac{7!}{4!3!} \\
&= \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} \\
&= 7 \cdot 5 \\
&= 35.
\end{aligned}$$

There are 35 ways to select the project group.

Example: In how many different ways can a group of 3 people out of a group of 7 people be chosen to work on a project if it has already been decided that a certain person must work on the project?

Note: *This is still a combination but the problem has been reduced to selecting 2 more people from the remaining 6 people.*

$$\begin{aligned}
\binom{6}{2} &= \frac{6!}{(6-2)!2!} \\
&= \frac{6!}{4!2!} \\
&= \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} \\
&= 3 \cdot 5 \\
&= 15.
\end{aligned}$$

There are 15 ways to select the project group.

Note: *Most calculators have a key for combinations.*