6.7. Probability.

The study of probability has increased in popularity over the years because of its wide range of practical applications.

In probability, each repetition of an experiment is called a **trial**, and the possible results of each trial are called **outcomes**. The set of all possible outcomes of a given experiment is called the **sample space** for the experiment. Any subset of the sample space is called an **event**.

Note: *The tossing of a coin over and over is called a trial. The landing of the coin on heads or tails is called an outcome. The results of tossing the coin any number of times is called the sample space. Any part of the sample space is called an event.*

Example: The sample space for the experiment of tossing a single die is {1, 2, 3, 4, 5, 6}. An event of this sample space is {4} or {2} or etc.

The notation P(E) is used to illustrate the **probability** of an event.

Probability of an Event E

In a sample space with equally likely outcomes, the **probability** of an event E, written P(E), is the ratio of the number of outcomes in the sample space S that belong to event E, n(E), to the total number of outcomes in sample space S, n(S).

That is,

$$P(E) = \frac{n(E)}{n(S)}.$$

Example: A single die is rolled.

(a) What is the probability that the number showing is odd?

S = {1, 2, 3, 4, 5, 6} and E = {1, 3, 5}.

n(S) = 6 and n(E) = 3.

$$\mathbf{P(numberodd) = \frac{3}{6} = \frac{1}{2}.}$$

(b) What is the probability that the number showing is greater than 2?

S = {1, 2, 3, 4, 5, 6} and E = {3, 4, 5, 6}.

n(S) = 6 and n(E) = 4.

$$\mathbf{P(number\,greater\,than\;2) = \frac{4}{6} = \frac{2}{3}.}$$

(c) What is the probability that the number showing is less than 7?

S = {1, 2, 3, 4, 5, 6} and E = {1, 2, 3, 4, 5, 6}.

n(S) = 6 and n(E) = 6.

$$\mathbf{P(number\,less\,than\;7) = \frac{6}{6} = 1.}$$

Note: *An event that has a probability of 1 is called a certain event. That is, the event is certain to occur.*

(d) What is the probability that the number showing is greater than 7?

S = {1, 2, 3, 4, 5, 6} and E = { }.

n(S) = 6 and n(E) = 0.

$$\mathbf{P(number\,greater\,than\;7) = \frac{0}{6} = 0.}$$

Note: *An event that has a probability of 0 is called an impossible event. That is, the event cannot occur.*

Example: A card is drawn from a standard deck of playing cards.

(a) What is the probability that the card drawn is a king?

Note: *There are 52 cards in a deck of cards, and 4 of these cards are kings.*

n(S) = 52 and n(E) = 4.

$$\mathbf{P(king) = \frac{4}{52} = \frac{1}{13}.}$$

(b) What is the probability that the card drawn is a face card?

Note: *There are 52 cards in a deck of cards, and 12 of these cards are face cards (4 kings, 4 queens, and 4 jacks).*

n(S) = 52 and n(E) = 12.

$$\mathbf{P(face\ card)} = \frac{12}{52} = \frac{3}{13}.$$

The set of all outcomes in the sample space that **do not** belong to event E is called the **complement** of E, written as E'.

Example: A single card is drawn from a deck of cards. If E is the event that the card is an ace, then E' is the event that the card is not an ace.

$$\mathbf{P(E) = P(an\ ace)} = \frac{4}{52} = \frac{1}{13} \ \ \textbf{and} \ \ \mathbf{P(E') = P(not\ an\ ace)} = \frac{48}{52} = \frac{12}{13}.$$

The probability of an event plus the probability of not an event equals 1. That is,

1. P(E) + P(E') = 1
2. P(E) = 1 – P(E')
3. P(E') = 1 – P(E)

Example: If the P(E) = 5/7, then P(E') = ?

P(E') = 1 – P(E)

P(E') = 1 – 5/7

P(E') = 2/7.

Sometimes probability statements are expressed in terms of **odds**, which is a comparison of P(E) with P(E').

The **odds in favor** of an event E are expressed as the ratio of P(E) to P(E') or as the fraction P(E)/P(E').

The **odds against** an event E are expressed as the ratio of P(E') to P(E) or as the fraction P(E')/P(E).

Example: If the probability of rain is 1/4, then what are the odds that it will rain?

If P(rain) = 1/4, then P(no rain) = 3/4.

The odds are P(rain) to P(no rain) = 1/4 to 3/4 = (1/4) ÷ (3/4) = 1/3 or 1 to 3.

Note: *The odds that it will not rain are 3 to 1 or 3/4 to 1/4.*

If the **odds favoring event E** are **m** to **n,** then

P(E) = m/(m+n)

Example: A shirt is selected at random from a dark closet containing 4 green shirts and 6 that are not green. Find the odds in favor of the green shirt being selected.

P(E) = P(green) = 4/10 = 2/5.

P(E') = 1 – P(E) = 1 – (2/5) = 3/5.

Odds in favor of green being selected are

P(E) to P(E') = 2/5 to 3/5 = (2/5) ÷ (3/5) = 2/3 or 2 to 3.

A **compound event** involves an alternative such as E or F, where E and F are events.

Two events E and F are **mutually exclusive** if they have no outcomes in common. Their intersection is the empty set. That is, $P(E \cap F) = 0$.

Probability of the Union of Two Events

If E and F are events in the same sample space, the probability of E or F occurring is given by $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

If E and F are mutually exclusive, then $P(E \cup F) = P(E) + P(F)$.

Example: A single card is drawn from a well-shuffled deck of 52 cards.

(a) What is the probability that the card is a king or diamond?

Note: *The events of drawing a king and drawing a diamond are not mutually exclusive since it is possible to draw the king of diamonds, which is an outcome satisfying both events.*

P(king or diamond) = P(king) + P(diamond) – P(king and diamond)

P(king or diamond) = (4/52) + (13/52) – (1/52)

P(king or diamond) = 16/52

P(king or diamond) = 4/13.

(b) What is the probability that the card is a four or a queen?

Note: *Drawing a 4 and drawing a queen are mutually exclusive events because it is impossible to draw one card that is both a 4 and a queen.*

P(4 or queen) = P(4) + P(queen)

P(4 or queen) = (4/52) + (4/52)

P(4 or queen) = 8/52

P(4 or queen) = 2/13.

Two events are said to be **independent** if the occurrence of one has no effect on the occurrence of the other.

Probability of Independent Events

If E and F are independent events, the probability that both E and F will occur is P(E and F) = P(E)• P(F).

Example: Assume that you draw two cards from a deck of 52 cards without replacement. Find the probability of drawing a heart and a spade.

Note: *Drawing a heart and drawing a spade are independent events since neither has an effect on the occurrence of the other.*

P(heart and spade) = P(heart) · P(spade)

P(heart and spade) = (13/52) · (13/51)

P(heart and spade) = 13/204.

Properties of Probability: For any events, E and F

1. $0 \leq P(E) \leq 1$
2. P(a certain event) = 1
3. P(an impossible event) = 0
4. P(E') = 1 – P(E)
5. P(E or F) = $P(E \cup F) = P(E) + P(F) - P(E \cap F)$